A new solution to an old problem: a temporary equilibrium version of the Ramsey model

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L'ottimo è nemico del bene
(Better is the enemy of good)

Italian proverb

Abstract

Convergence toward the optimal capital accumulation path in infinite horizon has always been tackled in the literature by means of the assumption that individuals (or a central planner) are able to select the unique convergent (saddle-)path among the infinitely many paths which satisfy the equimarginality condition of the intertemporal choice problem (the Euler's condition). This is tantamount to assuming that individuals have 'colossal' rational capabilities. Conversely, any minor deviation from the saddle-path would inevitably lead to a crash on a 0 per-capita consumption path. This paper aims to show that this contraposition is false. An asymptotic convergence result to the optimal equilibrium path will be obtained for an individual who plans myopically, that is, that optimizes his present and future consumption levels under a rudimentary hypothesis about future savings. He then partially re-adjusts his choices in each subsequent period, like people normally do. A similar result was already proved by the author for the central planner problem. In this paper, a 'market' solution is provided, following a temporary equilibrium approach *à la* Hicks.

Keywords. Optimal capital accumulation; Ramsey-Cass-Koopmans model, saddle-path (in)stability; myopic behaviour; temporary equilibrium.
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1 Introduction

A wide range of theoretical and empirical problems has been faced in the literature of the last 40-50 years on the basis of the analytical apparatus of intertemporal choices. The first authors who considered the problem of intertemporal choices are most likely von Böhm-Bawerk (1889) and Wicksell (1901). Yet, the milestone in this literature was presented by Ramsey (1928), who explicitly faced the problem of identifying the optimal level of savings, on the basis of a suggestion by Keynes who clearly had in mind the costs (and the benefits) of saving (see Ramsey (1928, p. 545)). Ramsey's analysis has reached contemporaneous analysis through the vulgate provided by Cass (1965) and Koopmans (1964), known as the Cass-Koopmans-Ramsey model. A standard formulation of this model is available in the majority of post-graduate textbooks of macroeconomics and growth.[1] The intertemporal choice problem is here formulated on an infinite horizon through an 'optimal control' problem (Ramsey originally formulated his analysis in terms of calculus of variations). Formally, the necessary conditions that identify the optimal path include a 'transversality' condition, since the optimal path takes the form of a *saddle* path. Given an initial level of capital per-worker, the transversality condition identifies the initial value of per-worker consumption needed to 'drop' the individual on the *unique* path which converges toward the long-run equilibrium. All the other initial levels of per-worker consumption bring the system along paths that sooner or later would lead to a 0 per-worker consumption. From an economic stance, this amounts to assuming that the consumer (or the central planner, if the problem is formulated in normative terms): i) calculates the infinitely-many paths which satisfy Euler's condition, that is, the marginality equalities between the intertemporal rate of substitution and the interest factor (or, the marginal productivity factor in the normative problem), and ii) selects among the (infinitely-many) paths the unique path which satisfies the transversality condition.[2] Should the consumer not hit 'the right' level of optimal consumption, the subsequent consumption levels deduced from Euler's condition would lead him to deviate definitively from the long-run equilibrium path.

A quite peculiar argument is often invoked to justify the assumption that individuals satisfy the transversality condition. The saddle path—which is unstable from the mathematical point of view—is seen as the unique possi-

[1] See, for example, Blanchard and Fischer (1989), Azariadis (1993), Barro and Sala-i-Martin (1995), Romer (1996), to name a few.

[2] In the case of the problem set on a *finite* horizon, the logic for finding the solution is more explicit: given the level of capital which must be left in the last period, one calculates the optimal consumption levels of the various periods by backward induction, i.e. starting from the last period. In the infinite horizon problem the adoption of the transversality condition replicates a similar argument: among the infinitely many paths, it selects the unique path which keeps consumption *positive* in the long-run.

bility for rational agents to coordinate among themselves. In a stable node, or in a focus, where all paths converge to the steady state equilibrium, agents would be unable to coordinate on the same path. On the contrary, when there exists a *unique* convergent path—like in the case of the saddle-path—rational agents know that the only way to avoid crashing on a zero-consumption path is to 'jump' *squarely* on it, thus engendering the convergence to the steady state of the system (on this, see Begg (1982, pp. 31-41)). This interpretation of the transversality condition requires a tremendous 'amount of rationality' by individuals (not surprisingly 'saddle-path stability' is at the basis of the strongest notion of rational expectations: the assumption of perfect foresight). (TO BE VERIFIED) Conversely, should the individual fail to identify the initial value of the 'jump' variable, the entire economy would be led irreparably away from the convergent path.

We will see in this paper that this contraposition is false. We will prove the opposite, that convergence to the steady state equilibrium can be obtained without assuming perfect foresight. In order to prove this, we will consider an individual who plans myopically: he optimizes his present and future consumption levels under a rudimentary hypothesis about future savings. Then, as time goes by, he re-adjusts his past choices in each subsequent period, like people normally do.

Part of the results here presented have been the object of a previous investigation (see Bellino (2013)), where the optimal path of the system is approached by a central planner in a similar way. In this paper I will present the 'market side' of the process. Clearly, this view will lead us from the intertemporal notion of equilibrium, inherent in the Cass-Koopmans-Ramsey model, to a *temporary* notion of equilibrium as described by Hicks (1939).

2 Description of the economic system

Consider an economic system where only one commodity is produced, consumed and employed jointly with labour as a capital good in its own production; the capital good depreciates at the rate $\mu \in [0,1]$. Time is considered a discrete variable, making it easier to analyse the evolution of the system as a sequence of events.[3] We define 'period t' the half-open time interval $[t, t+1)$ between dates t and $t+1$. Consumers live forever, and they are all equal. We can thus study the behaviour of the representative consumer. To simplify, let us suppose that the population remains constant. Consumer's preferences have a cardinal representation,[4] being described by a utility functional, $U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\theta}\right)^t u(c_t)$, which is constituted by the sum of discounted

[3]For discrete time versions of the Ramsey model see, for example, Azariadis (1993, chs. 7 and 13), or Stockey and Lucas (1989, ch. 2).

[4]See Koopmans (1965, section I); see also Hicks (1965, ch. XXI, in particular pp. 256-7 and Appendix E).

utilities achieved in each period, $u(c_t)$, where c_t is the consumption level in period t, $\frac{1}{1+\theta}$ is the discount factor of the future utility, and $\theta > 0$ is the rate of time preference. We suppose that $u : \mathbf{R}^+ \mapsto \mathbf{R}$ is a twice continuously differentiable, increasing and concave function; hence $u'(c_t) > 0$ and $u''(c_t) < 0$. For simplicity, let's assume also $\lim_{c_t \to 0^+} u'(c_t) = +\infty$.

The technology of the representative firm is represented by a continuously differentiable, homogeneous of the first degree production function, $F(K, L)$, defined for $K \geq 0$ and $L \geq 0$; as usual, $F_K > 0$, $F_L > 0$, $F_{KK} < 0$ and $F_{LL} < 0$. Let

$$r := Q/P \quad \text{and} \quad w := W/P,$$

where Q is the rent price of the good (to be employed as capital good), P is the unit price of the good, and W is the nominal wage rate. Hence, r is the rate of return of capital and w is the real wage rate.

3 The behaviour of the representative firm

Each (representative) firm maximizes extra-profits in each period;

$$\max_{K,L} \Pi = P \cdot F(K, L) - \mu PK - QK - WL, \tag{1}$$

where μ is the annual depreciation rate of capital. The first order conditions of (1) are

$$P \cdot F_K - \mu P = Q \tag{2a}$$
$$P \cdot F_L = W. \tag{2b}$$

As $F(K, L)$ is homogeneous of the first degree, $F(K, L)/L \equiv f(k)$, where $k = K/L$, $F_K \equiv f'(k)$ and, thanks to Euler's theorem we have $F_L \equiv f(k) - kf'(k)$. Thus, the first order condition (2) can be written as

$$f'(k) - \mu = r \tag{3a}$$
$$f(k) - kf'(k) = w. \tag{3b}$$

Suppose that when $t = 0$, the capital labour ratio k_0 is given, i.e.

$$k_0 = \bar{k}_0 < k^*, \tag{4}$$

so that, r_0 and w_0 are determined by (3):

$$\bar{r}_0 = f'(\bar{k}_0) - \mu \tag{5a}$$
$$\bar{w}_0 = f(\bar{k}_0) - \bar{k}_0 f'(\bar{k}_0). \tag{5b}$$

4 The behaviour of the representative consumer

In an intertemporal setting, the identification of an optimal consumption path requires that in the first period, $t = 0$, the consumer chooses the present consumption level *and all future consumption levels*. Here we propose an alternative to facing this problem, which is still based on 'rationality' arguments, but it does not require rationality that extends in such detail over each period of the infinite horizon.

Rather than mapping the 'realistic' behaviour of the representative consumer, our purpose here is to show that a 'small amount' of rationality is sufficient to channel the system towards its long-run path. We can thus focus on a simple way of facing future consumption choices. We will suppose that the consumer chooses current consumption and savings under the (provisional) assumption of zero net savings in all future periods. This conventional assumption provides us a way to weighting the costs and benefits of present savings in a simple manner. In fact, the assumption of zero future net savings (provisionally) puts the system on a steady path. This allows us to optimize the benefits of current savings on future utility without the need to choose *at present* the savings levels of each future period. But, since the assumption of zero future net savings is provisional, nothing prevents the consumer from relaxing it in each future period, should he find it convenient to do so (and he would).

4.1 Consumer decision for period $[0,1)$

Let

$$a_\tau = k_\tau - b_\tau, \qquad \tau = 0, 1, 2, \cdots \tag{6}$$

the amount of net activity owned by the consumer in period τ and b_τ the debt of the consumer. Given $\bar{r}_0$ and $\bar{w}_0$, for any given r_1 and w_1 the consumer chooses his consumption path by solving:[5]

$$\max U_0 = u(c_0) + \frac{u(c_1)}{1+\theta} + \frac{u(c_2)}{(1+\theta)^2} + \frac{u(c_3)}{(1+\theta)^3} + \cdots \tag{7a}$$

$$\text{s.v.} \qquad c_0 = \bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0 \tag{7b}$$

$$c_\tau = w_\tau + r_\tau a_1, \quad \tau = 1, 2, 3, \ldots \tag{7c}$$

As it appears from the set of constraints, consumers need to know future rental rates of capital, r_τ, and future wage rates w_τ, for $\tau = 1, 2, 3, \ldots$. The simplest expectation they can express is that the set of prices determined by the market for the next period, i.e. r_1 and w_1, will remain *constant* during the entire infinite future—even if they may differ from the present levels, $\bar{r}_0$ and $\bar{w}_0$: this is consistent with the assumption of a *stationary* set of future individual consumption and production plans.[6] This means that a decision

[5] Constraints (7b) and (7c) are written in terms of *relative* prices.

[6] In this regard Hicks wrote:

> '[a] stationary state is in full equilibrium, not merely when demands equal supplies at the currently established prices, but also when the same prices continue to rule at all dates—when prices are constant over time': Hicks (1939, p. 132).

Again,

> 'If plans are mostly of a fairly stationary type, so that most people are planning to

will be taken on the basis of the assumptions that

$$r_\tau = r_1 \quad \text{and} \quad w_\tau = w_1, \quad \tau = 1, 2, 3, \ldots, \tag{8}$$

and that consumers will consider prices r_1 and w_1 as *parameters*. Thanks to (7b), (7c) and (8) the objective function can be re-written as follows:

$$\begin{aligned} U_0 = u(\bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0) + \frac{u(w_1 + r_1 a_1)}{1+\theta} + \frac{u(w_1 + r_1 a_1)}{(1+\theta)^2} + \\ + \frac{u(w_1 + r_1 a_1)}{(1+\theta)^3} + \cdots = u(\bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0) + \frac{u(w_1 + r_1 a_1)}{\theta}, \end{aligned}$$

and the consumer's program becomes

$$\max_{a_1} \; u(\bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0) + \frac{u(w_1 + r_1 a_1)}{\theta}. \tag{9}$$

The first order condition of (9) is

$$\frac{\mathrm{d}U_0}{\mathrm{d}a_1} = 0, \quad \Leftrightarrow \quad u'(\bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0)(-1) + \frac{u'(w_1 + r_1 a_1)}{\theta} r_1 = 0. \tag{10}$$

that is,

$$u'(\bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0) = \frac{u'(w_1 + r_1 a_1)}{\theta} r_1. \tag{11}$$

The second order condition of (9) is

$$\frac{\mathrm{d}^2 U_0}{\mathrm{d}a_1^2} < 0, \quad \Leftrightarrow \quad u''(\bar{w}_0 + \bar{r}_0 a_0 - a_1 + a_0) + \frac{u''(w_1 + r_1 a_1)}{\theta} r_1^2 < 0, \tag{12}$$

which is always satisfied as $u''(\cdot) < 0$.

4.2 Temporary equilibrium for period $[0, 1)$

Since consumers are equal, their choices will be identical. Though each consumer can freely borrow and lend, no consumer will actually lend or borrow in equilibrium (if someone borrows someone else must lend; but this would contradict the constraint that consumers make the same choices). Thus, in equilibrium,

$$b_\tau = 0, \quad \tau = 1, 2, 3, \ldots,$$

and by definition (6) we have

$$a_\tau = k_\tau, \quad \tau = 0, 1. \tag{13}$$

Observe that this is a *characteristic of equilibrium*, not an *a priori* constraint imposed on the maximization problem.

buy and sell much the same quantities in future periods as in the current period, not much disequilibrium due to inconsistency will arise, so long as they merely expect a continuance of current prices': Hicks (1939, p. 136)

In particular,

$$a_0 = k_0 = \bar{k}_0, \tag{14}$$

Thanks to (3) written for $\tau = 0$ and $\tau = 1$, and thanks to (13), the first order condition (11) can be written as:

$$u'[f(\bar{k}_0) - \mu\bar{k}_0 - (k_1 - \bar{k}_0)] = u'[f(k_1) - \mu k_1]\frac{f'(k_1) - \mu}{\theta}. \tag{W0}$$

(W0) is an equation in k_1; it is a particular case of equation (Wt) (see Section 5 below) where parameter k_t is fixed at $k_t = \bar{k}_0$. Hence, by applying Lemma 1 below, (W0) has a unique solution, $k_1^\bullet$, such that

$$\bar{k}_0 < k_1^\bullet < k^*. \tag{15}$$

After substituting the equilibrium level of the capital/labour ratio of period $[0,1)$ into (3) we obtain the equilibrium levels of the remaining variables:

$$r_1^\diamond = f'(k_1^\bullet) - \mu \tag{16a}$$
$$w_1^\diamond = f(k_1^\bullet) - k_1^\bullet f'(k_1^\bullet). \tag{16b}$$

4.3 Consumer decision and temporary equilibrium for period $[1,2)$

On the basis of what has been planned in period $[0,1)$, at the beginning of period [1,2) consumers could consume

$$c_\tau = w_1^\diamond + r_1^\diamond k_1^\diamond = f(k_1^\bullet) - \mu k_1^\bullet, \qquad \tau = 1, 2, 3, \ldots.$$

for all future periods. In this case the rental rate of capital and the wage rate would remain exactly at levels (16). Nevertheless, they may wish to revise their previously selected consumption path. Analogously to what was done in period $[0,1)$, their revised choice can be found by solving the following problem:

$$\max U_1 = u(c_1) + \frac{u(c_2)}{1+\theta} + \frac{u(c_3)}{(1+\theta)^2} + \frac{u(c_4)}{(1+\theta)^3} + \cdots \tag{17}$$
$$\text{s.v.} \quad c_1 = w_1^\diamond + r_1^\diamond a_1 - a_2 + a_1 \tag{18}$$
$$c_\tau = w_2 + r_2 a_2, \quad \tau = 2, 3, 4, \ldots \tag{19}$$

Observe that as soon as consumers revise their choices, their steady expectation about future prices is replaced by another set of wages and profit rates, w_2 and r_2, different from the wage rate and the profit rate of period 1, but still constant from period 2 onwards. This is because consumers are planning a path which will be in a steady state from period 2 onwards.

After substituting the constraints, we obtain the following maximization problem:

$$\max_{a_2} \; u(w_1^\diamond + r_1^\diamond a_1^\diamond - a_2 + a_1^\diamond) + \frac{u(w_2 + r_2 a_2)}{1+\theta} + \\ + \frac{u(w_2 + r_2 a_2)}{(1+\theta)^2} + \frac{u(w_2 + r_2 a_2)}{(1+\theta)^3} + \cdots$$

that is,

$$\max_{a_2} \; u(w_1^\diamond + r_1^\diamond a_1^\diamond - a_2 + a_1^\diamond) + \frac{u(w_2 + r_2 a_2)}{\theta}, \tag{20}$$

whose first order condition is

$$u'(w_1^\diamond + r_1^\diamond a_1^\diamond - a_2 + a_1^\diamond) = \frac{u'(w_2 + r_2 a_2)}{\theta} r_2. \tag{21}$$

For the same reasons seen before (all consumers are equal: there are no loans or borrowings in equilibrium), we have

$$a_2 = k_2. \tag{22}$$

After substituting (3) written for $\tau = 1$ and $\tau = 2$, and equalities (13) and (22) into (21) we obtain

$$u'[f(k_1^\bullet) - \mu k_1^\bullet - (k_2 - k_1^\bullet)] = u'[f(k_2) - \mu k_2] \frac{f'(k_2) - \mu}{\theta}. \tag{W1}$$

(W1) is a particular case of equation (Wt) (see Section (5)), where parameter k_1 is fixed at $k_1^\bullet$. By applying Lemma 1 below, (W1) has a unique solution, $k_2^\bullet$, such that

$$k_1^\bullet < k_2^\bullet < k^*. \tag{23}$$

After substituting the equilibrium level of the capital/labour ratio of period $[1,2)$ into (3) the equilibrium levels of the remaining variables are obtained:

$$r_2^\diamond = f'(k_2^\bullet) - \mu \tag{24a}$$

$$w_2^\diamond = f(k_2^\bullet) - k_2^\bullet f'(k_2^\bullet). \tag{24b}$$

Remark As in equilibrium consumers do not borrow or lend among themselves ($b_\tau = 0$), there is no necessity to impose a transversality condition (or a no-Ponzi game condition) in the consumer's problem.

4.4 Consumer decision and temporary equilibrium for period $[t, t+1)$

In general, for any period $[t, t+1)$, and for any given initial level of capital/labour ratio, k_t, on the basis of the consumers' optimal behaviour and of the market temporary equilibrium conditions, we deduce that the capital/labour ratio for period $t+1$ is the solution of the following equation:[7]

$$u'[f(k_t) - \mu k_t - (k_{t+1} - k_t)] = \frac{u'[f(k_{t+1}) - \mu k_{t+1}]}{\theta}[f'(k_{t+1}) - \mu]. \qquad (\mathrm{W}t)$$

The sequence of equilibria thus described are *temporary* equilibria, in the sense described by Hicks (1939, chapter X), that is, market clearing equilibria in the current market based on a set of future plans taken by the representative consumer. In each period t the consumer makes his optimal, current, choice assuming a stationary behaviour for production and income for all future periods. Once $t+1$ is reached, he perceives that the assumed stationary behaviour is not his optimal choice *for period* $t+1$.[8] He thus revises his choice for that period on the basis of a stationary income and consumption path assumed for the future ($t+2$, $t+3$, etc.). The actual path of the economy is represented by the set of current market clearing equilibria obtained in the manner described above.

5 Convergence to the Ramsey modified golden rule

In this section we will study the analytical properties of sequence $\{k_t^\bullet\}_{t=1}^{\infty}$, generated by equation (Wt).[9]

Proposition 1. *The capital/labour ratio $k = k^*$, solution of the Ramsey modified golden rule,*

$$f'(k^*) = \theta + \mu, \qquad (25)$$

is the unique steady state of sequence $\{k_t^\bullet\}_{t=1}^{\infty}$.

Proof. A *steady state* of $\{k_t^\bullet\}_{t=1}^{\infty}$ is a value of k such that $k_t = k_{t+1} = k$. Substituting into (Wt) we get: $u'[f(k) - \mu k - (k-k)] = \frac{u'[f(k)-\mu k]}{\theta}[f'(k) - \mu]$, that is, $1 = [f'(k) - \mu]/\theta$, which coincides with (25), whose unique solution is $k = k^*$. Let

$$g_{k_t}(k_{t+1}) := u'[(1-\mu)k_t + f(k_t) - k_{t+1}]$$

and

$$h(k_{t+1}) := \frac{u'[f(k_{t+1}) - \mu k_{t+1}]}{\theta}[f'(k_{t+1}) - \mu].$$

g is a function of k_{t+1} parametrized by k_t.

[7] Equation (Wt) coincides with the equation describing the centralized solution of a planner who optimizes sequentially, in a way similar to that adopted here by the representative consumer (see Bellino (2013).

[8] According to Hicks's (1939, p. 134) classification, an incorrect forecast of its own wants was the cause originating this disequilibrium.

[9] The proofs here presented follow closely those provided in Bellino (2013) for the planner's solution. They are reproduced here for convenience.

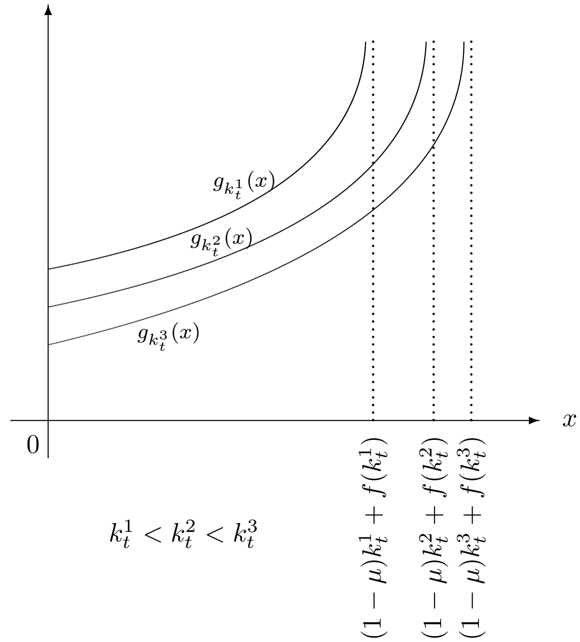

Figure 1: Curves $g_{k_t}(x)$

Properties of g Parameter k_t defines a sheaf of curves. Each of these curves is defined, continuous and strictly increasing for $k_{t+1} \in G_{k_t} = [0, (1-\mu)k_t + f(k_t)]$ (as u is decreasing). In the first quadrant, each of these curves has a finite and positive interception with the vertical axis, $u[(1-\mu)k_t + f(k_t)]$, and a vertical asymptote given by $k_{t+1} = (1-\mu)k_t + f(k_t)$. When parameter k_t increases, the interception with the vertical axis decreases, the abscissa of the vertical asymptote increases, and curve $g_{k_t}(\cdot)$ shifts downward, that is,

$$g_{k_t}(k) > g_{k_{t+1}}(k) \quad \text{if} \quad k_t < k_{t+1} \tag{26}$$

for those k where both are defined. Hence, curves $g_{k_t}(\cdot)$ never intersect themselves; they appear as in Figure 1.

Properties of h Function $h(k_{t+1})$ is defined where $f(k_{t+1}) - \mu k_{t+1} > 0$, that is, for $0 < k_{t+1} < \tilde{k}$, where $\tilde{k}$ is that level of k defined by $f(\tilde{k}) = \mu\tilde{k}$ which makes the net product equal to zero. Moreover,

$$\lim_{k_{t+1} \to 0^+} h(k_{t+1}) = +\infty \tag{27}$$

$$\lim_{k_{t+1} \to \tilde{k}^-} h(k_{t+1}) = -\infty \tag{28}$$

as $f'(\tilde{k}) - \mu < 0$. Define k_g as that level of k for which

$$f'(k_g) = \mu : \tag{29}$$

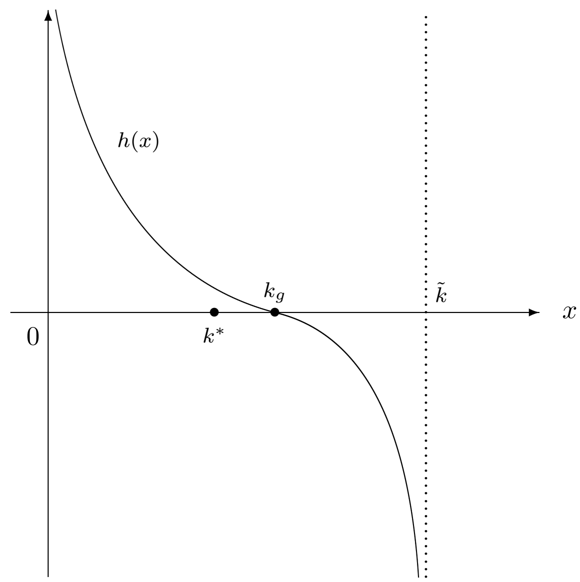

Figure 2: Curve $h(x)$

it is the so called golden rule capital/labour ratio: see Barro and Sala-i-Martin (1995, ch. 1). As f is decreasing, from (25) and (29) we deduce

$$k^* < k_g. \tag{30}$$

We have:

$$h(k_g) = 0 \tag{31}$$

and

$$\frac{\mathrm{d}h}{\mathrm{d}k_{t+1}} < 0 \tag{32}$$

as $u'' < 0$ and $f'' < 0$ in $0 < k_{t+1} < \tilde{k}$. Curve $h(k_{t+1})$ appears as in Figure 2.

Lemma 1. *Given $k_t \in (0, k^*)$:*

1. *there exists a unique $k^\bullet_{t+1} \in (0, \hat{k})$ which solves (Wt), where $\hat{k} = \min[(1-\mu)k_t + f(k_t), k_g]$, that is, there exists a unique $k^\bullet_{t+1}$ which solves (Wt) on the interval where both $g_{k_t}(x)$ and $h(x)$ are defined and positive;*
2. *$k^\bullet_{t+1} > k_t$;*
3. *$k^\bullet_{t+1} < k^*$.*

Proof. 1. Consider equation $g_{k_t}(k_{t+1}) = h(k_{t+1})$ on the restricted domain $k_{t+1} \in [0, \hat{k}]$. For $k_{t+1} \to 0^+$ we have $g_{k_t}(0^+) = g_{k_t}(0) = u'[(1-\mu)k_t + f(k_t)]$; hence

$$0 < g_{k_t}(0^+) < +\infty. \tag{33}$$

Moreover, (27) means

$$h(0^+) = +\infty. \tag{34}$$

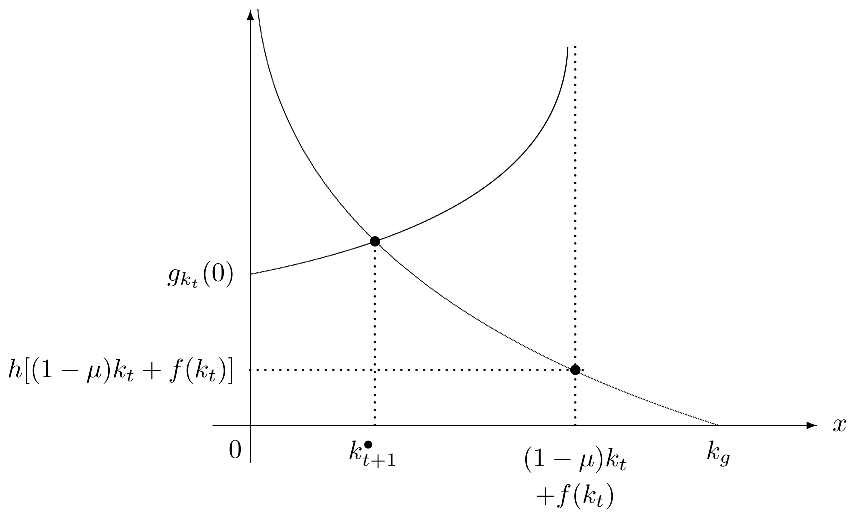

Figure 3: Case 1a: $(1-\mu)k_t + f(k_t) < k_g$

Hence, by (33) and (34) it follows that

$$g_{k_t}(0^+) < h(0^+). \tag{35}$$

Since $k_{t+1} = (1-\mu)k_t + f(k_t)$ is the vertical asymptote of $g_{k_t}(k_{t+1})$, we have

$$g_{k_t}\{[(1-\mu)k_t + f(k_t)]^-\} = +\infty. \tag{36}$$

In order to compare g_{k_t} and h at the other estreme of the domain, $\hat{k}$, three cases must be distinguished:

(a) If

$$(1-\mu)k_t + f(k_t) < k_g, \tag{37}$$

then $\hat{k} = (1-\mu)k_t + f(k_t)$ and curves $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ appear as in Figure 3. As $k_t > 0$ and from (37) we see that $h(x)$ is finite and positive at $k_{t+1} = (1-\mu)k_t + f(k_t)$, that is,

$$h[(1-\mu)k_t + f(k_t)] < \infty. \tag{38}$$

Hence by (36) and (38) we obtain

$$g_{k_t}\{[(1-\mu)k_t + f(k_t)]^-\} > h[(1-\mu)k_t + f(k_t)]. \tag{39}$$

By continuity and thanks to inequalities (35) and (39), we can conclude that there exists a unique

$$k^\bullet_{t+1} \in (0, (1-\mu)k_t + f(k_t)), \quad \text{that is,} \quad k^\bullet_{t+1} \in (0, \hat{k}) \tag{40}$$

which satisfies (Wt) (see Figure 3).

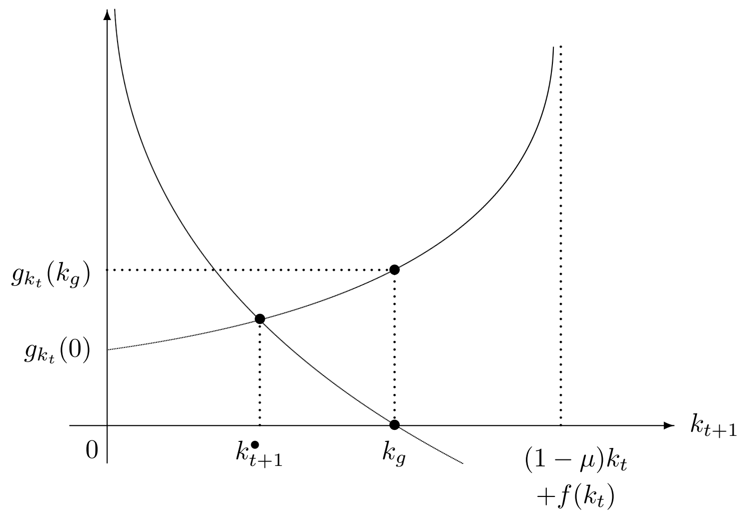

Figure 4: Case 1b: $k_g < (1-\mu)k_t + f(k_t)$

(b) If

$$k_g < (1-\mu)k_t + f(k_t), \tag{41}$$

then $\hat{k} = k_g$ and curves $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ appear as in Figure 4. Thanks to (41) we deduce that $g_{k_t}(k_{t+1})$ is finite and positive at $k_{t+1} = k_g$, that is

$$0 < g_{k_t}(k_g) < +\infty. \tag{42}$$

On the other hand (31) gives us

$$h(k_g) = 0. \tag{31$'$}$$

Hence by combining (42) and (31′) it follows that

$$g_{k_t}(k_g) > h(k_g). \tag{43}$$

By continuity and thanks to inequalities (35) and (43), we can conclude that there exists a unique

$$k^\bullet_{t+1} \in (0, k_g), \quad \text{that is,} \quad k^\bullet_{t+1} \in (0, \hat{k}) \tag{44}$$

which satisfies (Wt) (see Figure 4).

(c) If

$$k_g = (1-\mu)k_t + f(k_t), \tag{45}$$

then $\hat{k} = k_g = (1-\mu)k_t + f(k_t)$ and curves $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ appear as in Figure 5. In this case

$$g_{k_t}(\hat{k}^-) \equiv g_{k_t}\{[(1-\mu)k_t + f(k_t)]^-\} = +\infty \tag{46}$$

and

$$h(\hat{k}) \equiv h(k_g) = 0. \tag{47}$$

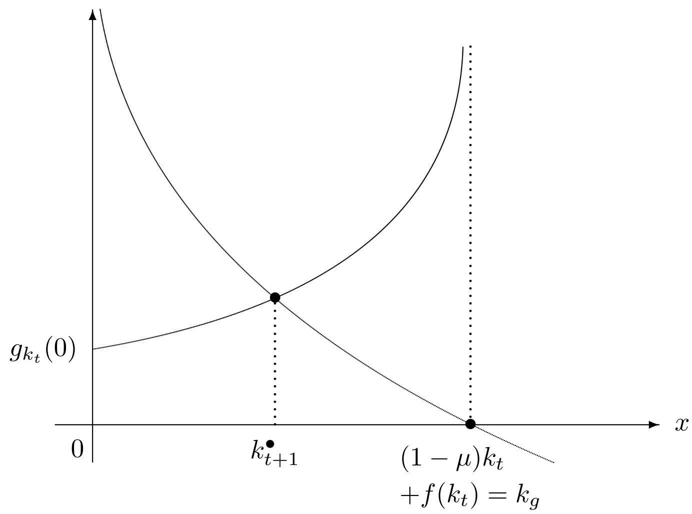

Figure 5: Case 1c: $k_g = (1-\mu)k_t + f(k_t)$

Hence by (46) and (47) it follows that

$$g_{k_t}(\hat{k}) > h(\hat{k}). \tag{48}$$

By continuity and thanks to inequalities (35) and (48), we can conclude that there exists a unique

$$k^{\bullet}_{t+1} \in (0, k_g), \quad \text{that is,} \quad k^{\bullet}_{t+1} \in (0, \hat{k}) \tag{49}$$

which satisfies (Wt) (see Figure 5).

2. Evaluate functions $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ at $k_{t+1} = k_t$:

$$\begin{aligned} g_{k_t}(k_t) &= u'[(1-\mu)k_t + f(k_t) - k_t] = u'[f(k_t) - \mu k_t] \\ h(k_t) &= \frac{u'[f(k_t) - \mu k_t]}{\theta}[f'(k_t) - \mu] \\ \text{hence} \quad g_{k_t}(k_t) &< h(k_t) \text{ as } \frac{f'(k_t) - \mu}{\theta} > 1 \text{ for } k_t < k^*. \end{aligned}$$

Curves $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ appear as in Figure 6: hence the solution $k^{\bullet}_{t+1}$ of (Wt) must thus lie *on the right* of k_t.

3. Draw curves $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ on the same graph (see Figure 7). Two cases must be distinguished.

   (i) If $(1-\mu)k_t + f(k_t) \leq k^*$, due to (30) we are under case 1a considered in the proof of this Lemma. Thus $k^{\bullet}_{t+1} < (1-\mu)k_t + f(k_t)$; hence $k^{\bullet}_{t+1} < k^*$ follows (see Figure 7((i)).

   (ii) If $(1-\mu)k_t + f(k_t) > k^*$, evaluate $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ at $k_{t+1} = k^*$:

$$\begin{aligned} g_{k_t}(k^*) &= u'[f(k_t) - \mu k_t - (k^* - k_t)] \\ h(k^*) &= \frac{u'[f(k^*) - \mu k^*]}{\theta}[f'(k^*) - \mu] = u'[f(k^*) - \mu k^*] \quad \text{due to (25)} \end{aligned} \quad .$$

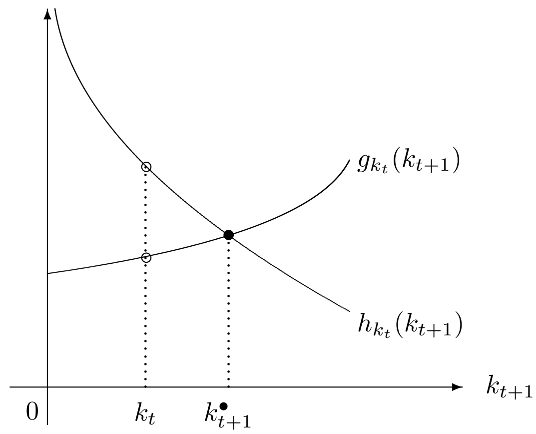

Figure 6: Lemma 1-items 1 and 2

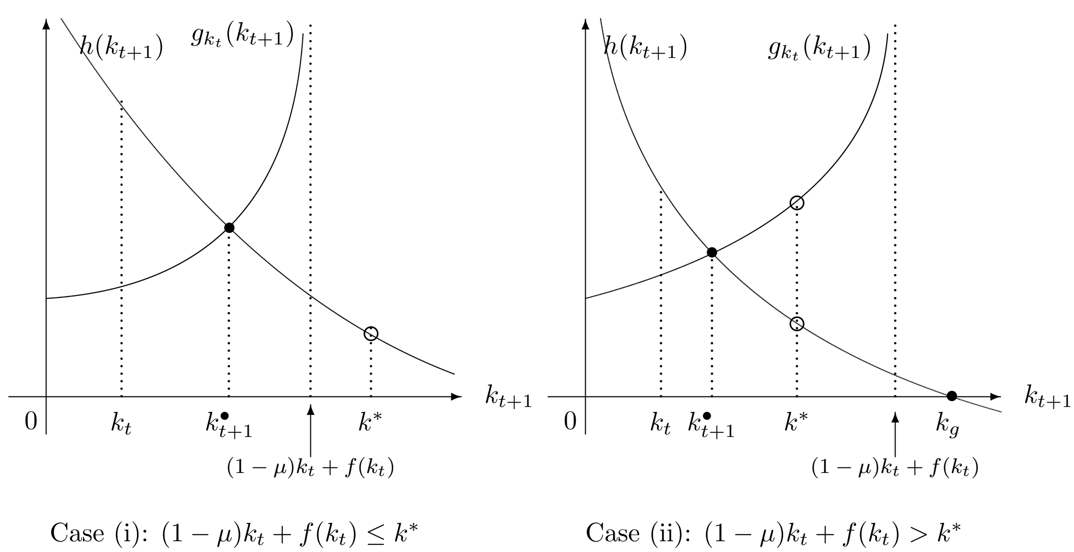

Figure 7: Lemma 1-item 3

As $k_t < k^*$, then $f(k_t) - \mu k_t - (k^* - k_t) < f(k^*) - \mu k^*$; as u' is decreasing, then $g_{k_t}(k^*) > h(k^*)$. Curves $g_{k_t}(k_{t+1})$ and $h(k_{t+1})$ appear thus as in Figure 7((ii)); hence the solution $k^\bullet_{t+1}$ of (Wt) must lie *on the left* of k^*.

This completes the proof. □

Now, we are going to show that if $k_0 = k^*$, equation (Wt) defines a constant sequence: $k_t = k^*$, $t = 1, 2, 3, ...$

Lemma 2. *If $k_t = k^*$, there exists a unique $k^\bullet_{t+1}$ which solves (Wt): it is $k^\bullet_{t+1} = k^*$.*

Proof. Thanks to equation (25) it is straightforward to verify that equation

$$g_{k^*}(k_{t+1}) = h(k_{t+1}) \tag{W*}$$

is satisfied by $k^\bullet_{t+1} = k^*$. Moreover, the left-hand member of (W*) is a monotonically increasing function of k_{t+1}, while the right-hand member of (W*) is a monotonically decreasing function of k_{t+1}. Hence k^* is the unique solution of (W*). □

Lemmas 1 and 2 entail that, given $k_0 \in (0, k^*]$, a sequence $\{k^\bullet_{t+1}\}_{t=0}^\infty$ contained in $(0, k^*]$ is univocally defined by recurrence by equation (Wt).

Lemma 3. *$k = k^*$ is the unique steady-state of sequence $\{k^\bullet_{t+1}\}_{t=0}^\infty$.*

Proof. A steady-state of $\{k^\bullet_{t+1}\}_{t=0}^\infty$ is a value of the capital/labour ratio such that $k_t = k_{t+1} = k$. Substituting it into (Wt), we obtain $u'[f(k) - \mu k - (k - k)] = \frac{u'[f(k)-\mu k]}{\theta}[f'(k) - \mu]$ which, after simplification, reduces to, $[f(k)\mu]/\theta = 1$, whose unique solution is $k = k^*$ (see equation (25)). □

Proposition 2. *If $k_0 = \bar{k}_0 < k^*$, the sequence $\{k^\bullet_t\}_{t=1}^\infty$ of capital/labour ratios converges monotonically to the steady state k^* defined by the Ramsey modified golden rule* (25)*.*

Proof. By Lemma 1, if $\bar{k}_0 < k^*$ sequence $\{k^\bullet_t\}_{t=1}^\infty$ is *monotonically increasing* (thanks to item 2) and *upper bounded* by k^* (thanks to item 3). Hence it must converge to some k',

$$\lim_{t\to\infty} k^\bullet_t = k'. \tag{50}$$

In order to prove that $k = k^*$ observe that, by definition, the elements $k^\bullet_t$ of the sequence satisfy equations (Wt). Consider the limit for $t \to \infty$ of (Wt):

$$\lim_{t\to\infty} u'[f(k_t) - \mu k_t - (k_{t+1} - k_t)] = \lim_{t\to\infty} \frac{u'[f(k_{t+1}) - \mu k_{t+1}]}{\theta}[f'(k_{t+1}) - \mu].$$

Thanks to the continuity of functions u', f and f' we can write

$$u'\left[f\left(\lim_{t\to\infty} k^\bullet_t\right) - \mu \lim_{t\to\infty} k^\bullet_t - \left(\lim_{t\to\infty} k^\bullet_{t+1} - \lim_{t\to\infty} k^\bullet_t\right)\right] = \frac{u'\left[f\left(\lim_{t\to\infty} k^\bullet_{t+1}\right) - \mu \lim_{t\to\infty} k^\bullet_{t+1}\right]}{\theta}\left[f'\left(\lim_{t\to\infty} k^\bullet_{t+1}\right) - \mu\right].$$

which, thanks to (50), can be written as

$$u'[f(k') - \mu k' - (k' - k')] = \frac{u'[f(k') - \mu k']}{\theta}[f'(k') - \mu].$$

After simplification, this equation in k' reduces to $[f'(k)\mu]/\theta = 1$, whose unique solution is $k = k^*$ (see equation (25)). This completes the proof. □

6 Concluding remarks

The convergence of the sequence of temporary market equilibria to the Ramsey steady state path provides us with an insight on the 'amount of rationality' needed to drive an economic system *à la* Ramsey towards its long-run equilibrium. The Ramsey problem of identifying the optimal consumption/savings path is usually faced, in the Cass-Koopmans version, by means of an intertemporal equilibrium approach. In the initial period, the representative consumer must solve his trade-off between consumption and savings for the present period as well as for the (infinitely many) subsequent periods. For this purpose, he must exclude the infinitely-many paths which diverge from the saddle-path (in analytical terms, he must selects that unique consumption-savings path which satisfies the transversality condition from the infinitely-many paths solving the Eulero equation). Under the intertemporal equilibrium setting, the representative consumer must display an enormous computing ability. In other terms, he is not allowed to divert from the saddle path, not even by a little. A small deviation would in fact sooner or later entail crashing on a zero-consumption path.

In the present paper, the Ramsey problem has been settled in the *temporary equilibrium framework*. As we have seen, the representative consumer optimizes his present and future consumption levels using a rudimentary hypothesis on future savings. He then re-adjusts his past choices in each subsequent period, like people normally do. This short-term optimizing behaviour allows individuals to move towards Ramsey's steady state equilibrium.

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