On the Complex Management of Common-Pool Resources: A Coalition Theory Approach

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Abstract. In this work, I present an overview of the common pool resource, which includes its definition, classification, and nature. Given that it is a complex problem involving several variables that favor cooperation, I use a coalition approach to study some of them. Namely, communication, group size, and homogeneity of members of a group. I draw from a baseline model of appropriation that reflects the dilemma of the common pool resources, which is a strategic game. Then using this model I study conditions under which forming a group may be beneficial. Next, I address the gains of cooperation, for which, I transform the game into a partition function game and verify that fulfills some results in the existing literature. Thus, this function is symmetric, the grand coalition is a efficient partition, and it has a γ-core. Besides that, I apply a game called the payoff sharing game to study formation of coalition structures. This last part is still in progress.

Keywords: Common Pool Resources (CPR) · Cooperation · Coalitions · Groups · Appropriation · γ-Core

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1 Introduction

From the beginning of time, humans (homo sapiens) have been regarded as the most complex form of life on earth. It is well known that what distinguishes them from other creatures is their capacity to reason over their own conduct. In this sense, human behavior could be said to be driven by reasoning. However, it is not as simple as one thinks. Humans also possess instincts and other characteristics such as feelings, which are actually inherent. Some of their acts are driven by these. Sometimes following instincts can lead to behavior that can jeopardize others. However -as I said- instincts have been part of humans from the very beginning and they have been fundamental for human evolution. Take for instance the survival instinct, dated back to prehistory. It brought humans to hunt animals to survive. But it was the capacity of reasoning that led to cooperation, so they could thrive and perpetuate the species. Of course, it is clear that they hunted mainly to meet their basic needs, but they knew, for instance, that in order to hunt a mammoth, it was necessary for collaboration of more than one member of the tribe.

Humans are complex beings by themselves, but they are social cooperative species as well. From many years ago, they have learned to live together. And it is in this process they have been also able to stay in contact with nature. They have known how to take advantage of their reasoning to make the most of this interaction considering or not the consequences. On the other hand, nature provides resources that humans take for their livelihood, benefit, or to create other kind of resources. Typically such resources possess two crucial attributes that affect the behavior of the individuals who exploit them. That is to say, what one individual uses from them subtracts from another individual from its use, but at the same time, it is highly difficult to exclude others from using them. A high degree of subtractability of use together with a high degree of difficulty of excluding potential beneficiaries led to social dilemmas in terms of cooperation. In the literature, these resources that share the attribute of substractabiliy with private goods and difficulty of exclusion with public goods are termed common-pool resources, and they are a paradigm that puzzles the human behavior; the unselfishness over the immediate material self-interest, reflection over instincts and other factors that enter into the picture.

More precisely, the joint use of common-pool resources, such as fisheries, forests, lakes, and groundwater basins may lead toward an over-exploitation when involved people are pursuing their own interests. There are diverse elements at play when it is about the common-pool resources issue, which are in fact what makes it complex to account for successful organization by all involved individuals. Nonetheless, the core of the problem is captured in the literature as a social dilemma. And not surprisingly, non-cooperative game theory comes into play as a useful starting point. In other words, owing to the free overuse of a common-pool resource and the fact that individually optimal behavior produces a socially and individually suboptimal result, the individuals end up depleting the resource. Such situation generates a problem that can be formalized as social dilemma game.[1] Under this approach, individuals are rational and try to maximize their utilities which predicts a non-cooperative outcome. Individuals profit by exploiting the resources as much as they can without caring about others. Nevertheless, Gardner, Ostrom, and James M Walker (1990), Ostrom et al. (1994) challenge this argument. They provide evidence from field studies and experiments where people actually steer away from the individual outcome. Although it is admitted that cases of unsuccessful cooperation exist, Janssen and Ostrom (2006). And actually situations of both type of outcomes are still evident nowadays. Such is the case of some communities in Mexico.

On one hand, there is the recurring problem of water management that face some dwellers of a touristic village called *Tamul* in the state of San Luis Potosí. This small village is well known for its natural landscapes and waterfalls. People there live off the land and tourism. One of the main activities related to the land is the sugar cane growth, cutting practices, milling and raw sugar processing.

1. In a social dilemma game there is a strong interdependence between individual outcomes and other outcomes, people, by pursuing immediate-self interest, can harm their owns group's interest Liebrand (1983).

Not surprisingly, the use and management of water is a big issue there. Mainly because the water that is used to irrigate the crops comes from the rivers and falls, which, at the same time, are the major tourist attraction. In this place, since there is no way of reaching the big fall by walking but by boating, there is a group of boatmen that make a living of this. Over the last three years, this fall has run out of water periodically because of the excessive extraction for irrigation of the great cane fields. This situation has led the boatmen committee to lodge a complaint with local authorities (State Water Commission). Both parties came to an agreement. During holidays, the Water Commission agreed to cut off the water supply for irrigation proposes, whereas during other periods they determined what they call *tandeo* (distribution of irrigation water by turns). Nonetheless, this has not been respected in full, and not authorized water diversions have been spotted. Thus the common pool resource problem is still proving hard to manage. Even in presence of a an agreement, it is not fulfilled.

On the other hand, in the State of Oaxaca, again in Mexico, there is a unique legally recognized program which enables its municipalities[2] being ruled by traditional governance practices. The so-called *usos y costumbres* program coexists with formal institutions in certain municipalities with high indigenous populations. Among other aspects, this traditional governance institution has a system called *cargo* or *tequio* to solve collective actions and sanction those who refuse to cooperate in activities for the common good. They call *tequio* to a charge or an assignment for a member or group of members of the community. Within this system, it is of particular interest how they manage common resources such as the water. As it is mentioned in Diaz-Cayeros, Magaloni, and Euler (2009), indigenous members form The Water Committee is in charge of monitoring water use and well's reserves, punishing wasteful practices as well as fixing water pipes. Some of the punishments include fines, cut-offs of water supply or even physical punishment. Such practices, however, differ from other indigenous communities[3]. In this connection Magaloni, Dıaz-Cayeros, and Euler (2018) demonstrate that communities ruled by traditional governance practices offer more effective provision of local public goods (including common pool resources, such as water from wells) than equally poor communities ruled by political parties.

As I said, the issue of the common pool resources is not uncomplicated, and many factors come into play. Studying them separately is useful to grasp the problem and to devise solutions. Cooperative game theory and formation of coalitions approach include factors that the standard non cooperative game theory disregards. The possibility of involved individuals to coalesce, to communicate, and the size of groups are of particular interest in this work, since they are identified by empirical researchers as common factors that promote cooperation

2. In Mexico a *free municipality*, idea arisen from the Mexican revolution, refers to the basic entity of its political-administrative division. Each of their municipalities possesses full autonomy trough its own legislative and executive power.

3. Indigenous population is diverse, so the *Uses* and *costumbres* program recognizes this diversity and confers constitutionally traditional governance for each of the diverse communities.

and self-govern in common pool resources. In this sense, the work of Meinhardt (2012) is already an advancement. He sets the common-pool resources problem under the domains of cooperative game theory. He finds interesting results in terms of understanding the incentives to cooperation towards -expressed in cooperative game theory terms- the grand coalition; notwithstanding, he does not consider cases in which people prefer to cooperate in groups that are smaller than the grand coalition. Therefore, my work proposes to explore this. The rationale lies on answering the question of what coalition patterns can be found in a common pool resources setting. The paper proceeds as follows, in section two I present the framework of the common pool resources and an overview of the common factors associated to cooperation. Section three deals with a baseline model of appropriation. Here I study, without introducing cooperative game theory tools yet, how grouping may be beneficial for the actors of the model. Also I study issues related to the size of population. Then in section four, I transform the strategic game model introduced in section three into a model in partition function form, so that I study some of the results of Parkash (2019) in relation to my model; this part is still in progress. Also, in this section I present some field cases where coalition formation have come up.

2 Common-Pool Resources

In his influential work, Samuelson (1954) divides goods into two kinds, **pure private goods** and **pure public goods**. According to him, the former is both excludeble and rivalrous whereas the latter is not. That is, under this classification, if the public good is supplied, no consumer can be prevented from consuming it, and the consumption of it by one consumer does not limit the quantity available for consumption by others. Nevertheless, such definitions were rejected by Ostrom (2010). She states that the Samuelson's twofold classification is consist with a dual view of the organizational forms of society. First, that the market is the optimal institution for the production and exchange of private goods. And second, that the government is seen as the owner of a property organized by a public hierarchy. Then she goes deeper into this simplistic dual division and proposes, together with her collaborators[4], additional modifications. First, to replace the term "**rivalry** of consumption" with "**subtractability** of use." Second, to conceptualize subtractability of use and excludability to vary from low to high rather than characterizing them as either present or absent. Third, overtly to add a very important fourth type of good **-common-pool resources-** that shares the attribute of **subctractability** with private goods and difficulty of **exclusion** with public goods. And forth, to change the name of "club good" to "toll good" since many goods that share these characteristics are provided by small scale public as well private associations. In this sense, following Ostrom (2008), "common-pool resources are seen as sufficiently large that it is difficult, but not impossible, to define recognized users and exclude other users altogether.

4. See Ostrom and Ostrom (1999)

Further, each person's use of such resources subtracts benefits that others might enjoy." This new taxonomic modifications can be arrayed in Table 1, which for clarity contains some examples.

Moreover, in the literature, the common-pool resources are further classified into two types. Namely, open-access resources and common-property resources, in opposition to private property resources. The former are such that property rights are held by community of individuals and may include the government and non-government organizations, and their use can be regulated in a variety of ways by a variety of institutions, Common and Stagl (2005). Following Tietenberg and Lewis (2018), Some common pool resources may admit property rights. However, such rights may be costly to enforce, so they are not exercised. In contrast, in open access resources not everything is subject to property rights. Here no one owns or exercises control over the resource. Anyone can enter freely to exploit the resource in a *first-come, first-served basis* . And no individual or group has the capacity or the legal power to restrict access. Such characteristic promotes a *use it or lose i*t situation. Open-access resources have given rise to what has become known popularly as the "tragedy of the commons" —see Hardin (1968) and Lloyd (1833). In a contrasting manner, open-access resources may be over-exploited but common property resources need not suffer overuse and their allocation can be regulated in a way that avoids the tragedy. Here it is worth quoiting Elionor Ostrom's distinction between that tragedy and the problem of commons:

> [T]he problem is that people can overuse, they [the sources] can be destroyed, and it is a big challenge to try to figure out how to avoid it. That is a problem, that is real. The tragedy is the way he [Hardin (1968)] expresses it, they cannot, ever, solve it. That is different.—It is inevitable and unconquerable. That is why he called it a tragedy. They were trapped... and the only way out was some external government coming in or diving it up into small chunks and everyone owing their own... Ostrom (2009).

In essence, the difference lies in two aspects. From the outset, it is not merely a tragedy; instead, it is a problem that needs not be neither ineluctable nor ineludible. Second, there are different ways of avoiding it, one of them could be —although not necessarily the best—external entities. Thus studying what and how could be the best way of preventing the problem is a big concern and a matter of debate. In fact, as Janssen and Ostrom (2006) highlight, there are examples of both successful and unsuccessful efforts to govern and manage common-pool resources by governments, communal groups, cooperatives, voluntary associations, and private individuals of firms Berkes (1989), Bromley et al. (1992), Katar et al. (1994), Singh, Ballabh, et al. (1996). That said, notice again that given the nature of the open accesses resources, the "tragedy" may emerge eventually. This does not mean that just open accesses resources are endangered by overuse. Every common-pool resource can face deterioration by unsustainable use, but the latter ones are more vulnerable.

		Subtractability of Use	
		High	**Low**
Difficulty of excluding potential beneficiaries	**High**	Common-pool resources: groundwater basins, lakes, irrigation systems, fisheries, forests.	Public goods: peace and security of a community, national defense, knowledge, fire protection, weather forecasts.
	Low	Private goods: food, clothing, automobiles.	Toll goods: theaters, private clubs, daycare centers.

Table 1. Taken from Ostrom (2010).

2.1 Appropriation and The Nature of Common-Pool Resources

In the same line of Plott and Meyer (1975), the process of withdrawing units from any kind of common pool resource is termed appropriation, and thus the person who withdraws such units from it is, accordingly, an appropriator. Following Ostrom et al. (1994), the problems that appropriators face can be studied separately. They cluster them into two types, appropriation and provision. In the former, there is an assumed production relationship between yield and level of inputs. Here the problem to be solved is how to allocate equitably that yield, or input activities to achieve it. Appropriation problems deal with the allocation of the units of extraction of the resource as a flow. More specifically, the problem has to do with the following aspects. One, the quantity of resource units to be appropriated, or the establishment of the efficient level of input resources necessary for obtaining that flow of units of the resource. Second, timing and location of appropriation as well as the technology for appropriation. On the other hand, the provision problems deal with the creation, maintenance, and the improvement of productive capabilities of the resource as well as avoiding its depletion or destruction. Here the units of use of the resource are seen as stock. Notice that in real world situations a common pool resource may be complex and exhibit problems of appropriation and provision. However, it is useful to study both problems separately. In this work, I focus on the former.

In this respect, according to Gardner, Ostrom, and James M. Walker (1990), there are four necessary conditions to produce a common-pool resources dilemma, and more notably, to distinguish it from a simple common-pool situation. To begin with, *resource unit substractability* is strongly linked to the definition of a common-pool resource. This condition tells, as it was already mentioned, that a resource unit extracted, harvested or withdrawn by one individual makes it unavailable for another one. Such extracted unit —the argument goes —is possible since the resource provides a never-ending flow of units over time as long as the degree of appropriateness do not outweigh the degree of replacement or regeneration of it. Also, in cases where there is not a replacement, the resource is exhaustible, and then one cannot talk about a flow but just of a stock of it that is gradually depleted. The second condition is the *existence of multiple appropriators*, the resource is withdrawn by more than one person or teams of individuals. Third, *sub-optimal outcomes*, which means that the appropriators' strategies yield sub-optimal outcomes given a configuration of their own attributes, the market conditions, technology, and the physical system. Forth is *constitutional feasible alternatives*. Here the authors touch upon the existence of a set of coordinated strategies that are more efficient than current decisions, and that they are constitutionally feasible given the current institutional and constitutional arrangements. Within this condition, in turn, I find that a sufficient condition for such set of feasible alternatives is the existence of a Pareto-optimal set of coordinates strategies that are individually advantageous to the involved appropriators.

As the reader can infer now, the definition of a common-pool resources together with conditions one and two lead to what is called common-pool resources situations. And whereas conditions three and four are necessary for a dilemma. There is not a dilemma if sub-optimal outcomes does not come up for at least a setting made of the factors of condition three. Also, a dilemma do not manifest when a set of constitutional feasible strategies do not produce a better outcome for appropriators.

2.2 Common Variables Involved in Common Pool Resources

From the point of view of the experimental psychological research, Kopelman, Weber, and Messick (2002) identify nine variables that influence cooperation in common dilemmas, to wit, social motives, gender, payoff structure, uncertainty, power and status, group size, communication, causes, and frames. In turn, they categorize such variables into individual differences (stable personal traits such as social motives and gender) and situation factors (the environment). The latter category is further differentiated into task structure (which orderly is composed by the decision structure and the social structure) and the perception of the tasks or perceptual factors (causes and frames). Within the decision structure there are the variables of payoff structure and uncertainty, whereas the social structure category includes the variables power and status, communication, and group size. These two last variables are particularly interesting for the purposes of my work. The size of the group, and the ability of people to communicate with one another are fundamental elements highly related to the limitations of the standard game theory.

Ostrom (2015) shows cases of the study of common-pool resources use, especially of successful groups avoiding the Nash outcome. One of the crucial conditions she detects, under which coordination succeeds, has to do with the number of individuals involved. Also Ostrom, Walker, and Gardner (1992) discuss a series of experiments approaching issues of individual behavior under common-pool situations. They set up experiments so as to gain a general explanation over how communication and punishing mechanisms on the group level influence individual behavior. Once they introduce these elements into the mix, they observe that the outcomes of the experiments generate behavior clearly inconsistent to the predictions of non-cooperative game theory. Moreover, when individuals are allowed to communicate with each other, they achieve significant improvements from group interactions even in the absence of punishing mechanisms.

In this connection, group size and communication under a common-pool resource context have been the object of investigation. In Kopelman, Weber, and Messick (2002) there is an interesting discussion of the experimental commons dilemmas literature regarding these two elements. According to them, two explanations of the effect of communication on cooperation, provided by Dawes, Van de Kragt, and Orbell (1990), are salient. First, group discussion enhances group identity or solidarity, and second, group discussion elicits commitments to cooperate. On the other hand, the group size issue has been highly a matter of debate. So far, there is no consensus on whether small size groups achieve more cooperative outcomes than the larger ones. The discussion presented in Kopelman, Weber, and Messick (2002) is not conclusive. In this line, Allison, McQueen, and Schaerfl (1992) explains that small groups are more motivated to divide resources equally than are members of large groups, whereas Agrawal and Goyal (2001) suggest that there is a curvilinear relationship between group size and successful collective action.

On the other hand, Janssen and Ostrom (2006) highlight nine variables commonly found in empirical studies related to self-governed resource use. Namely, information about the condition of the resource and expected flow ofbenefits and costs are available at low cost to the participants; second, appropriators plan to live and work in the same area for a long time; third, they are highly dependent on the resource; forth, appropriators use collective-choice rules that fall between the extremes of unanimity or control by a few; fifth, the group using the resource is relatively stable; sixth, the size of the group is relatively small; seventh, the group is relatively homogeneous; eighth, participants have developed generalized norms of reciprocity and trust that can be used as initial social capital; and ninth, Participants can develop relatively accurate and low-cost monitoring and sanctioning arrangements.

3 Model Description

In this model, I study the problem of appropriation decision in a static environment with very basic rule settings. This model is taken from Falk, Fehr, Fischbacher, et al. (2002), which depicts the setting of the baseline common-pool resource experiments conducted by Walker, Gardner, and Ostrom (1990). The baseline game is as follows. Each appropritator i has an endowment of resources, w_i, which in the symmetric case is e for everyone. All n players in the group decide independently and simultaneously how much they want to invest in the CPR. Individuals $i's$ investment decision is denoted by x_i[5]. The investment decision causes a cost c per unit of investment but also yields a revenue. Although the cost is assumed to be independent of the decisions of the other group members, the revenue depends on the investment decisions of all players. More specifically, the total revenue of all players from the common-pool resource

5. One interpretation of investment in this setting could be the time dedicated to the common pool resource exploitation.

is given by $f(\sum x_j)$ where $\sum x_j$ is the amount of total investment. For low levels of it, $f(\sum x_j)$ is increasing in $\sum x_j$,but beyond a certain level, $f(\sum x_j)$ is decreasing in $\sum x_j$. An individual subject i receives a fraction of $f(\sum x_j)$ according to the individuals share in total investment $\dfrac{x_i}{\sum x_j}$. Thus, this model emulates an environment most closely parallel to that of a limit-access resource. Thus the total material payoff of i is given by:

$$u_i(x_i, x_{-i}) = e - cx_i + \left[\frac{x_i}{x(N)}\right] f(x(N)) \tag{1}$$

where $x_{-i} = (x_1, ...x_{i-1}, ..., x_n)$ and $x(N) = \sum_{i \in N} x_i$. Formally, f is strictly concave and assume that $f(0) = 0$ and $f'(0) > c$, and $f'(ne) < 0$. Initially the investment in the common-pool resource yields positive returns [$f'(0) > c$], but if the appropriators invest a sufficiently large number of resources, say $\hat{q}$, the outcome is detrimental [$f'(\hat{q}) < 0$]. The yield from the common-pool resource reaches a maximum net level when individuals invest some, but not all, of their endowment in that resource.

So individual i solves

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i, x_{-i}) \\ \text{s.t.} \quad & 0 \le x_i \le e, \; i = 1, \ldots, n. \end{aligned}$$

Following,Ostrom et al. (1994), suppose that x_i^* solves the constrained maximized problem, and that $u_i(x_1, \ldots x_i^*, \ldots x_n)$ is the maximal value. This gives one equation in n unknowns. Solve now for each individual i. Thus there are n equations in n unknowns. A solution to this system of equations is a Nash equilibrium.[6] Now, since this game is symmetric in terms of endowments, strategies, and payoff functions; the wearisome problem of solving n simultaneous equations in n unknowns can be circumvented.[7] Thus, it is enough to solve for one individual i knowing that each solution will be the same for all of them. Now, given the assumptions on f, and for large enough values of e, there is an interior solution that satisfies the first order condition,

$$-c + \frac{x_i}{x(N)} f'(x(N)) + \frac{x(N) - x_i}{(x(N))^2} f(x(N)) = 0. \tag{2}$$

6. At the Nash Equilibrium, all involved individuals maximize simultaneously their respective utility. To see this, suppose that long as all other individuals are maximizing at a Nash equilibrium, the problem that individual i faces becomes

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i^*, ..., x_i, ...x_n^*) \\ \text{s.t.} \quad & 0 \le x_i \le e, \; i = 1, \ldots, n. \end{aligned}$$

which is solved by x_i^*.

7. Finite symmetric games have symmetric equilibria Nash (1951).

Introduce the symmetry assumption, so $x(N)$ becomes nx_i^*. Plug it into (2) yields,

$$-c + \frac{1}{n} f'(nx_i^*) + \frac{n-1}{n^2 x_i^*} f(nx_i^*) = 0 \tag{3}$$

In a nutshell, the interpretations of this equilibrium are the standard ones. For a rational player the solution that maximizes the problem is unique, so different choices of x_i^* will be sub-optimal, and there are no incentives for a rational player to deviate from this outcome. Now, let me calculate the social optimum $x^*(N)$, which is the unique solution maximizing the expression below subject to the constraint $0 \leq x(N) \leq ne$.

$$\sum_{i \in N} u(x_i) = u(x(N)) = ne - cx(N) + f(x(N)) \tag{4}$$

so the first order condition is

$$-c + f'(x(N)) = 0 \tag{5}$$

The marginal cost equals the marginal return from the common-pool resource. It is the maximal yield that can be extracted from the resource in a single period (Ostrom et al. (1994)). Now comparing (5) and (3) the reader will realize that agent's equilibrium behavior is not collectively optimal. It can be observed as well that the interior solutions of both maximization problems do not depend on the endowments. However, if they are not sufficiency large, the latter claim does not hold, and the solution for the maximization problems would be such endowments. Also, the fact that the Nash equilibrium does not depend on e implies that it does not account for the potential pressure over the resource that high levels of endowments may generate.

Now, consider a specific form of the revenue function used by Walker, Gardner, and Ostrom (1990) in their experiments, which was based on Gordon (1954) classic model.

$$f(x(N)) = ax(N) - bx(N)^2$$

with $c < a = f'(0)$, and $f'(ne) = a - 2bne < 0$. Recall, each player is endowment with e, and the cost per unit of exploitation is c. Thus the payoff of individual i is the next.

$$u_i(x_i, x_{-i}) = e - cx_i + \left[\frac{x_i}{x(N)}\right] [ax(N) - bx(N)^2] \tag{6}$$

$$u_i(x_i, x_{-i}) = e + (a-c)x_i - x_i bx(N)$$

Say that $(a-c) = \alpha$, so

$$u_i(x_i, x_{-i}) = e + \alpha x_i - x_i bx(N) \tag{7}$$

Proposition 1 (Nash Equilibrium) *If every agent acts individually and makes her own best decision given of all other agents, the optimal allocation x_i given the allocations of the rest of the agents is the symmetric Nash Equilibrium stated by* $x_i^* = \dfrac{\alpha}{b(n+1)}$

Proof (Nash Equilibrium). Maximize (7) w.r.t to x_i and follow the procedure as equation (2) and (3). ■

Likewise,

Proposition 2 (Social Optimum) *If all involved agents act cooperatively the social optimum or the Pareto optimum allocation is given by* $\dfrac{\alpha}{2b}$

Proof. From (4) and (5) and given the function $f(x(N)) = ax(N) - bx(N)^2$, it follows that

$$-c + a - 2bx(N) = 0$$

$$\alpha - 2bx(N) = 0$$

$$x^*(N) = \frac{\alpha}{2b}$$

■

In a single period, this represents the maximal yield that can be extracted from the resource. More than that, the return decreases.

Now the payoff the players get as a group implementing the social optimum is

$$ne + \frac{\alpha^2}{4b} \tag{8}$$

whereas the payoff of the symmetric Nash equilibrium group investment, $\dfrac{n}{n+1}\left[\dfrac{\alpha}{b}\right]$, is

$$ne + \frac{\alpha^2}{b}\frac{n}{(n+1)^2} \tag{9}$$

Notice that the former is greater than the latter since $1/4 > \frac{n}{(n+1)^2}$ as long as $n > 1$. Also, when the group investment is twofold the social optimal, i.e. (α/b), the group payoff is just ne, which means that there is no return from the common pool resource. Moreover, this value is also reached when implementing the symmetric Nash equilibrium group investment, the number of involved individuals increases, $\lim_{n\to\infty}\left(\frac{n}{(n+1)}\ \frac{\alpha}{b}\right) = \frac{\alpha}{b}$.

Let me present now, for the sake of the argument, a numerical example[8] at works

8. Taken fromFalk, Fehr, Fischbacher, et al. (2002)

Example 1 *Say that $e = 10$ and $c = 5$, and the total revenue is given by*

$$f\left(\sum x_j\right) = 23 \sum x_j - 0.25 \left(\sum x_j\right)^2 \tag{10}$$

Thus the material payoffs are:

$$u_i = 10 - 5x_i + \left[\frac{x_i}{\sum x_j}\right] \left[23 \sum x_j - 0.25 \left(\sum x_j\right)^2\right] \tag{11}$$

$$u_i = 10 + 18x_i - 0.25x_i \sum x_j \tag{12}$$

Now, to find the optimal individual case, maximize equation (4) with respect to x_i . The first order condition is

$$18 - 0.5x_i - 0.25 \left(\sum x_i - x_i\right) \tag{13}$$

Again we have now n first-order conditions for the n individuals that would need solving, so introduce the symmetry assumption that $\sum x_i = nx_i^*$ and plug into (13) *to get*

$$18 - 0.5x_i^* - 0.25(nx_i^* - x_i^*) \tag{14}$$

$$x_i^* = \frac{18}{0.5 + 0.25(n-1)} = \frac{18}{0.25(n+1)}$$

Now, we can compare this equilibrium allocation to the investment that would maximize the overall group yield from private and collective investment. The overall output for the group is given by (wit x as the vector of the individual allocations to the CPR)

$$\pi(x) = 10n + 18 \left(\sum x_i\right) - 0.25 \left(\sum x_i\right)^2 \tag{15}$$

There is a unique solution maximizing this expression, that can be found from the first order condition,

$$18 - 0.5 \left(\sum x_i\right) = 0 \tag{16}$$

$$\sum x_i = 36$$

Comparing equation (14) and equation (16), I find that they yield different results. The individuals equilibrium behavior is not collectively optimal.

In the experiments mentioned in Ostrom (2010), the initial resource endowment were tokens that the subject could allocate to the common-pool resource. For their experiment they use eight individuals. Now, the game theoretic outcome involves substantial overuse of a resource while a much better outcome could be reached if the subjects were to reduce their joint allocation. The prediction of the non-cooperative game theory was that subjects would invest according to the Nash equilibrium —8 tokens each for a total of 64 tokens. However, subjects could earn considerable more if they reduced their allocation down to a total of 36 tokens in the resource. Observe the payoffs for both, group optimal investment and symmetric Nash equilibrium group investment:

- Group payoff under the NE $\sum u_i = 10(8) + 18(64) - 0.25(64^2) = 208$
- Group payoff under the social optimum: $\sum u_i = 10(8)+18(36)-0.25(36^2) = 404$

However, the result of those experiments lead that people move away from the individualistic outcome. In this line, many communities are able to spontaneously develop their own approaches to manage common-pool resources. See several cases in Ostrom (2015) where people craft arrangements in a fashion different from the standard predictions are presented. Now one way to conciliate theory and practice in this subject, at least partially, is to approach the problem under the scrutiny of groups and coalition theory. Formation of groups that act as a single entity might shed light on the coordination of players and overcome individualistic outcomes. In the following sections, I explore this idea.

3.1 Groups and Individual Behavior

Suppose there is a group S of players smaller than the whole community willing to reduce its investment in the common pool resource to the one predicted by the social optimum. This group raises awareness about the benefits of cooperation. This comes up as something that which happens of itself, without any coercion but the will. That means that its members are willing to reduce the exploitation of the resource while the others are acting individually. Recall that the social optimum is $x^*(N)$. Suppose that it can be split out by the number of individuals involved in the game, so $\dfrac{x^*(N)}{n} := x_i^{**}$, but that this share is used just for the people in the group. If it was the case, the utility of individuals of group S of implementing x_i^{**} while the rest remain implementing x_i^* of the original game, is the next one,

$$\sum_{i\in S} u_i(x_i^{**}, x_{-i}^*) = |S|e + \left[\sum_{i\in S} x_i^{**}\right]\left[\frac{f(\sum_{i\in S} x_i^{**} + \sum_{j\notin S} x_j^*)}{\sum_{i\in S} x_i^{**} + \sum_{j\notin S} x_j^*} - c\right] \tag{17}$$

contrasting with the utility that individuals of group S when its members and the others implement the original NE,

$$\sum_{i\in S} u_i(x_i^{*}, x_{-i}^*) = |S|e + \left[\sum_{i\in S} x_i^{*}\right]\left[\frac{f(\sum_{i\in S} x_i^{*} + \sum_{j\notin S} x_j^*)}{\sum_{i\in S} x_i^{*} + \sum_{j\notin S} x_j^*} - c\right] \tag{18}$$

where $|S|$ stands for the carnality of group S. Which will be willing to implement the social optimum if and only if

$$\sum_{i\in S} u_i(x_i^{**}, x_{-i}^*) \geqslant \sum_{i\in S} u_i(x_i^{*}, x_{-i}^*) \tag{19}$$

Now, since the individuals who join to the group S are exploiting the resource at a level proportional to the social optimum, and the other non-cooperative individuals are investing at levels according to the NE, we have that $\sum_{i\in S} x_i^{**} = |S|\left[\dfrac{x^*(N)}{n}\right]$. Thus inequality (19) becomes,

$$|S|\left[\frac{x^*(N)}{n}\right]\left[\frac{f\left(|S|\left[\frac{x^*(N)}{n}\right]+[n-|S|]\,x_j^*\right)}{|S|\left[\frac{x^*(N)}{n}\right]+[n-|S|]\,x_j^*}\right] \geqslant |S|x_i^*\left[\frac{f\left(|S|x_i^*+[n-|S|]x_j^*\right)}{|S|x_i^*+[n-|S|]x_j^*}\right] \tag{20}$$

After some algebra, (20) is expressed as

$$x^*(N)\left[\frac{f\left(\frac{|S|}{n}\left[x^*(N)-nx_j^*\right]+nx_j^*\right)}{\frac{|S|}{n}\left[x^*(N)-nx_j^*\right]+nx_j^*}\right] \geqslant f(nx_i^*) \tag{21}$$

Now I plug the function of Walker, Gardner, and Ostrom (1990) and study that the inequalities (19) trough (21), so I arrive at the following.

Proposition 3 *Given the number of individuals n involved in the common pool resource problem, forming a group S such that exploits the resource at levels dictated in proportion to the social optimum x^{**} yields a greater return for its members rather than not forming it as long as $|S|$ approaches to n and the non-members remain acting individually. Whenever $|S| \to n > 1$, cooperative individuals have incentives to form such group.*

Proof. The utility of the potential group S of implementing the social optimum while the other the NE is

$$\sum_{i\in S} u_i(x_i^{**}, x_{-i}^*) = |S|e + \frac{|S|}{n}\left(\frac{\alpha}{2b}\right)\left[\alpha - b\left[\frac{|S|}{n}\left(\frac{\alpha}{2b}\right) + (n-|S|)\frac{\alpha}{b(n+1)}\right]\right] \tag{22}$$

$$= |S|e + \frac{\alpha^2|S|\left(n|S|+2n-|S|\right)}{4b(n+1)n^2}$$

whereas the utility of coalition S of implementing the NE just like the others is

$$\sum_{i\in S} u_i(x_i^*, x_{-i}^*) = |S|e + |S|\left(\frac{\alpha}{b(n+1)}\right)\left[\alpha - b\left[\frac{|S|\alpha}{b(n+1)} + \frac{(n-|S|)\alpha}{b(n+1)}\right]\right] \tag{23}$$

$$= |S|e + \frac{|S|}{b}\left[\frac{\alpha}{(n+1)}\right]^2$$

Now take

$$\lim_{|S|\to n}\sum_{i\in S} u_i(x_i^{**}, x_{-i}^*) = ne + \frac{\alpha^2}{4b}$$

and

$$\lim_{|S|\to n}\sum_{i\in S}u_i(x_i^*,x_{-i}^*)=ne+\frac{n}{(n+1)^2}\frac{\alpha^2}{b}$$

so, as I showed before, the latter value is smaller than the former one. ∎

Remark 1 (Large n). The result holds as n gets large, and $|S|$ gets close to it.

$$\lim_{|S|\to n\to\infty}\sum_{i\in S}u_i(x_i^{**},x_{-i}^*)=\infty+\frac{\alpha^2}{4b}$$

and

$$\lim_{|S|\to n}\sum_{i\in S}u_i(x_i^*,x_{-i}^*)=\infty$$

In words, the group S has to be a little smaller than the entire population (understood as n) in order to be effective. If there is no way of forming a group in which all members be included, a smaller group will be enough. Notice however that if n gets large whereas the group $|S|$ remains with a fixed size, then the cooperative individuals (individuals interested in forming the group) will have no incentive to stay in, since for a larger number of n the return that the common-pool resource yields is virtually null for either cases (using the social optimum or the Nash equilibrium). In addition to that, there are as well other conditions related to the size of the group in which the formation of the group may not be robust.

Proposition 4 *For a given numbers of involved individuals, n, not too large, if $|S|=\dfrac{2n}{n+1}$ implementing the social optimum yields the same return than implementing the Nash equilibrium. But if $|S|$ is greater strictly than this fraction and n does not get large, there are incentives to form the group.*

Proof. See the appendix ∎

When the number of individuals n involved in the exploitation of the resource is not too large, it is sufficient that more than two people form a group to obtain better gains. In other terms, when $|S|$ makes around $\dfrac{2}{n+1}$ of n, forming a group in terms of payoffs is not too attractive. Notice, however, that both propositions three and four depend strongly upon of the size of n. This is in line with the current debate regarding the size of the group and cooperation in common pool resources. As it was already mentioned, when the size of the group is relatively small, cooperation, which translates into self-manage of the resource, is easier to achieve (Wilson and Thompson (1993), Franzen (1984), Fujiie, Hayami, and Kikuchi (2005)). Nevertheless, when the size of n is too big, I found that a group has to be of a size barely smaller than it, so that there are incentives to group (at least until a certain point). That means that the number of people interested in cooperating should be sufficiently big in a big population. Although, under big populations, it may happen that the common pool resource does not resist it even under self-managing. As I set forth before, the social optimum does not depend upon n, but it does when I split it out by the number of population and then I assign a correspond share of it to the group S. Thus the greater the size of the population and the greater the size of S the fewer the individuals are going to appropriate. In this sense for big populations exploiting at low levels yields very lower returns which implies that monetary inducements for cooperation vanish.

Now assume that the individuals interested in cooperating disregard the social optimum, but still are willing to form a group. Thus they decide to implement that level of appropriation investment that maximizes their joint utility taking as given the individual investment of the non cooperative individuals. In other words, assume that players who are not interested in cooperating chose to implement the level dictated by the Nash Equilibrium whereas the cooperative players joint to a group that implement a optimum group investment given the individualistic behavior of the others. This is going to happen if the following holds

$$\sum_{i\in S} u_i(x_i^{**}, x_{j\notin S}^*) \leq \sum_{i\in S} u_i(x_S^*, x_{j\notin S}^*) \tag{24}$$

where $x_S := \sum_{i\in S} x_i$ and $x_S^* \in \arg\max \sum_{i\in S} u_i(x_S, x_{j\notin S}^*)$, and $x_{j\notin S}^*$ is the individual NE investment decision of every one of the members outside the group S. Expressly, group S maximizes wrt to x_S given that the others are choosing individually their level of investment.

so,

$$\sum_{i\in S} u_i(x_S, x_{j\notin S}^*) := u_S(x_S, x_{j\notin S}^*) = |S|e - x_S \left[\alpha - b\left[x_S + (n-|S|)x_{j\notin S}^*)\right]\right] \tag{25}$$

Since $x_{j\notin S}^* = \frac{\alpha}{b(n+1)}$, thus

$$u_S(x_S, x_{j\notin S}^*) = |S|e + \alpha x_S - b x_S^2 - x_S(n-|S|)\frac{\alpha}{n+1} \tag{26}$$

Now, maximize (27) w.r.t. x_S. F.O.C. for an interior solution.

$$\alpha - 2bx_S - \frac{\alpha(n-|S|)}{(n+1)} = 0$$

thus,

$$x_S^* = \frac{\alpha}{2b}\left[\frac{1+|S|}{n+1}\right] \tag{27}$$

Plug (28) into (27),

$$u_S(x_S^*, x_{j\notin S}^*) = |S|e + \left(\frac{1}{b}\right)\left[\frac{\alpha(|S|+1)}{2(n+1)}\right]^2 \tag{28}$$

Thus I have that

Proposition 5 *For a n not too large, a group S smaller than it that acts as a single entity will prefer to exploit the resource at the level that maximizes the sum of the utilities of its members (group optimum) rather than the social optimum level given that the outside individuals act individually. But if n gets large, the associate return of both cases will tend towards the same return; although depending on $|S|$, it is either too great or just $|S|e$. When $|S|$ approaches n, the former case happens, but when it is fixed, the latter comes about.*

Proof. See Appendix ∎

Again, forming a group is advantageous for individuals with a sense of cooperation. In this context, the fact that they are able to coordinate and decide upon the level of appropriation brings about greater gains when implementing a group optimal level of appropriation. This argument, nevertheless, is untenable when the number of appropriators is great. In the extreme cases in which the number of individuals exploiting the resource is too large, the pressure over it is higher as well, so its return goes to zero. Thus the utility the players get by grouping is that of summing up their endowments, which does make sense, since the aim of jointing is to coordinate the management of the resource so that obtain greater gains. Then when a population is considerably large, a cooperative group is not binding or simply fizzle out.

So far I just studied how the formation of a group may be beneficial. However, studying the problem of common pool resources under this perspective implies studying how more solid and complex groups come about. In this sense, a deeper understanding on formation of groups in called. Properly said, a group bestowed on individual character of its own becomes now the subject of study. In the next lines I explain this insight better.

4 Coalitions and Cooperative Game Theory

As I mentioned before in section 2.2, there are common variables that helps to explain cooperative behavior. In this sense, cooperative game theory and coalition formation accounts communication mechanism, bargain, homogeneity of the participants, and group size within its framework. What follows now is the relation of common pool resources and coalitions through some cases of study. I present a cooperative model in partition function form derived from the strategic game of the common pool resources. In addition to that, in the last section there is a game of formation of coalitions, which I apply to the case of common pool game.

4.1 Coalitions and Common Pool Resources

Consider the case where players are able to coalesce in a sense beyond of a mere group. That is, again cooperation between them is permitted, and binding agreements can be implemented. That said, the analysis of common pool resource situations changes. The players are considered to negotiate, they can group now.

There are different possibilities in terms of what groups may come up with. The basic entities of study now are those groups, which more precisely in the literature are termed as *coalitions*. In this sense, players may be involved in a bargaining process which enables them to adopt binding agreements. If players perceive that by cooperating with other players they receive more than what they are able to get by themselves, they might want to enter into negotiations the latter ones. Otherwise, they pursue an alternative option. The result of such negotiation processes aim at some stable coalitions in which players have no incentives to deviate from an agreement. Thus the aim in this section is to approach the gains of cooperation in a common pool resources environment under this approach.

In this connection, the literature shows cases where it studied the effect of groups size in the management of common pool resources as well as cases where people actually have come up with coalitions. Wilson and Thompson (1993) study the reasons behind a breakdown in productivity of communally held Mexican lands called ejidos.[9] They attribute such reasons to a deterioration in property management at the community level. According to this work, rights, duties, functions, and obligations of individual herders had not been clearly specified or enforced by ejido authorities in that time. Nevertheless, *failure* of group management —they argue—had led to the formation of coalitions within smaller groups where cooperation is assured and benefits are enjoyed under severe ecological conditions. They call "compensating coalitions" of the ejidos in the sense that they recognize the failure of the ejido, and in response, they try to compensate it by forming a group with enough structure to make a collective decision that benefits its members. The uncertainly of others' behavior is reduced in these coalitions, which enables them to reach a level of cooperative individuals short of the full cooperation level.

Besides that, Perez-Verdin et al. (2009) conduct an empirical study in which they test the relationship between common-based property regimes and the conservation of natural resources. Specifically, they study the effect of group size and heterogeneity upon the performance of ejidos to protect their forest resources in northern Mexico. What they actually arrived at was that, in general, group size and heterogeneity had no significant effects on the presence of deforested conditions. According to them, deforestation is driven by resource-specific characteristics, such as location and soil productivity, not by ejidos' characteristics, such as total area or number of members. In this vein, Poteete and Ostrom (2004) approach the research of the International Forestry Resources and Institutions related to, among other aspects, the interrelations among group size, heterogeneity, and institutions. They posit that actually group size and some forms of collective action has to have a non-linear relationship.

9. An ejido combines communal ownership with individual use. It consists of cultivated land, pastureland, other uncultivated lands, and the *fundo legal* (town-site) Britannica (2011). The *ejidos* controls a substantial share of the Mexican agricultural land.

On the other hand, in the field example exposed in Gardner, Ostrom, and James M Walker (1990) concerning a fishery in Sri Lankan, in addition to the analysis of the dynamic adjustment from a partially solved common-pool resources dilemma to one of failure the authors describe, I want to highlight how in certain situations the formation of groups emerge as a way of managing the exploitation of a resource. Let me explain. People in this small fishing village used beach seines as a technology (to catch fishes), but as each net was expensive and at least eight men were needed to shot and draw it ashore, they decided to split the ownership of a single net into eight shares. Then they used approximately twenty jointly owned beach-seines. And each share was single-handedly worked by a fisher, and the catch was divided equally among the eight owners. Observe how in this case, factors such as the characteristics of the resource (size and boundlessness of it) as well as of the used technology (the size, weight and costs of the beach seine) led people to from groups to devise a way of exploitation collectively (at least until a certain point).

4.2 The Coalition Approach Model

Let me now introduce the cooperative model of the common pool resources problems of appropriation. As I said, the analysis changes slightly since the entity of study are coalitions. Notice that this does not mean I disregard cases in which individuals just want to remain single. Thus, I set the problem of the common pool resources into a particular form of cooperative games[10]: the partition function form. This form takes into account possible externalities that coalitions impose on each other (recall: what I subtract from the resource you can not). Basically, by setting the common pool resources issue under coalitional structures I explore the formation of coalitions and the allocation of coalition worth to its members. In this sense, I study situations in which extreme cases of cooperation (no one forms a coalition or all players join) may or not arise as well as intermediate cases.

4.3 The Partition Function and the γ-Core

Even when the reader is familiarized already with what a coalition is, I define it formally for the sake of the argument and give some clarifications of it.

Definition 1 (Coalitions) *Let N be, again, the finite set of players. Formally, a group of players $S \in N$ is called a coalition. Specifically, $\emptyset$ is denoted as the empty coalition and the player set N itself is denoted as the grand coalition. And the collection of all coalitions is denoted by the power set $\mathcal{N} := 2^N$*

10. A cooperative game is also called n-person transferable utility game, since it is assumed that there is a commodity, say money, that players in a coalition can freely transfer among themselves. This assumption implies that disregarding of how the coalition payoff is split out, its members enjoy the same utility, since the payoffs are given to coalitions and not for individual players.

According to Gilles (2010), a *coalition* has to be thought of in a broader sense. It has a purpose and is assumed to be able to formulate and execute collective action. This entails that the members of a coalitions are provided with a collective decision mechanism or a governance structure. Accordingly, players are allowed to plan, formulate and execute collective actions trough institution, behavioral norms, and communication structures. In this light, this argument ties together with the strands advanced in Ostrom (2010) and Ostrom (2002). These studies posit that participants involved in a common-pool resources situation do undertake efforts to design their own governance arrangements and that substantial empirical evidence underpins it. Here is why -as I see it- it is interesting to approach the common-pool resources issue under a coalition approach.

The Partition Function Form. Following Parkash (2019), given a partition of the total player set into coalitions, a partition function Thrall and Lucas (1963) assigns a payoff to each coalition in the partition. A strategic game can be converted into a partition function if each induced strategic game in which each coalition in the partition becomes one single player admits a *unique* Nash equilibrium. Th The common pool game actually fulfills this condition.

Formally, a set $P = S_1, ...S_m$ is a partition of N if $S_i \cap S_j = \emptyset$ for all $i, j \in 1, ...m$, $i \neq j$, and $\bigcup_{i=1}^{m} S_i = S$. The worth of coalition S_i is given by $v(S_i; P) \geq 0$, which denotes the Nash Equilibrium payoff of coalition S_i in the induced strategic game in which each coalition S_j, $j = 1, ..., m$, becomes one single player, i.e. within the coalition the individual strategies are selected so as to maximize the payoff of the coalition: the sum of the payoff of its members. Then, (N, v) denotes the partition function form of the strategic game (N, X, u) where $X = \Pi_{i \in N} X_i$ is the set of strategies profiles, X_i is the strategy set of player i, so $X_i = [0, w_i]$ with $w_i > 0$ being the endowments for each players, and $u = (u_i, \ldots, u_n)$ is the vector of payoff functions, and u_i is the payoff function player i. A strategic profile is denoted by $x = (x_1, \ldots, x_n) \in X$. Here I am going to deal again with the symmetric case, so $X_i = \{x_i \in \mathbb{R}_+ : 0 \leq x_i \leq e\}$.

In line with Parkash (2019), I have now a partition function generated from the underlying common pool resources strategic game, which implies two things. First, the grand coalition has at its disposal a broader set of strategies, that is $[0, ne]$, which means that it can choose at least the same strategies as the players can in any game induced by a partition. In this sense, the grand coalition is an efficient coalition, formally $v(N; \{N\}) > \sum_{S_i \in P}^{m} v(S_i; P), \forall P = \{S_1, \ldots, S_m\} \neq \{N\}$. Second, given a partition, the members of a coalition has the possibility to decide on not to form it. How? Notice that each player of a coalition has the strategy set $[0, e]$, thus the coalition strategy set is $[0, |S|e]$, which means that the members of this coalition can choose the same strategies as they were singleton in the partition. Whenever the players choose such strategies given the strategies of the others, the coalition S_i does not form.

The γ-Core One important concept in cooperative games is that one of the core. It assigns to each cooperative game the set of payoffs that no coalition can improve upon by any coaliton. If a payoff does not belong to the core, one should not expect to see it as the prediction of the theory if there is full cooperation Serrano (2015). So far in the literature related to common pool games in characteristic function have studied classical core concept such as the α-core and β-core Meinhardt (2012), but other solution concepts have not explored in common pool resources in partition function form. Here, I start out my analysis with the one proposed by Parkash 2019, it is the γ-core.

Definition 2 (Feasible payoff) *Given a partition function game (N, v), a payoff vector $(z_i, ..., z_n)$ is feasible if $\sum_{i \in N} z_i = v(N; N)$.*

A feasible payoff is the division of the grand coalition.

Definition 3 (γ-core) *The γ-core of a partition function (N, v) is the set of feasible payoff vectors $(z_1, ..., z_n)$ such that $\sum_{i \in S} z_i \geq v\left(S; \{S, [N \setminus S]\}\right)$ for all $S \subset N$.*

where $[N]$ and $[N \setminus S]$ indicate the finest partitions of the N and $N \setminus S$ respectively.

Now since I assumed that the common pool game is symmetric in the sense that individuals have the same costs of extraction, same endowments, and same utility functions; this implies that its partition function form game is also symmetric, and the the worth of a coalition will depend on its cardinality. In other words, given a partition, two or more coalitions with the same number of members each will get the same worth.

Definition 4 (A Symmetric Partition Function Game) *A partition function game is symmetric if for every partition $P = \{S_i, \ldots, S_m\}$, $|S_i| = |S_j|$, then $v(S_i; P) = v(S_i; P)$*

Moreover, the common pool game partition function form (keeping the same quadratic function of section three) belongs to a particular class of symmetric partition function games; that is to say, those ones where the grand coalition is an efficient partition, and larger coalitions in each partition have lower per-members payoffs. To see the latter claim, say that I have a partition $P = \{S_i, \ldots, S_m\}$, and that coalition $|S_i| = |S_j|$, $i, j \in 1, 2, \ldots, m$ so the worth of both coalitions under the common pool game studied is

$$v(S_i; P) = |S_i|e + \frac{\alpha^2}{b(|P| + 1)^2} = |S_j|e + \frac{\alpha^2}{b(|P| + 1)^2} = v(S_j : P)$$

but if $|S_i| < |S_j|$, thus the lower per-member payoffs are such that

$$\frac{v(S_i; P)}{|S_i|} > \frac{v(S_j; P)}{|S_j|}.$$

Given that the common pool game in partition function form is symmetric and fulfills the above, I know from Parkash (2019)[11] that it has a non-empty γ-core as long as the grand coalition is an efficient partition. And that the feasible payoff vector with equal shares belongs to the γ-core and that the largest coalition in each partition is worse-off relative to this feasible payoff vector. I verify that effectively such claims are met.

The feasible payoff vector with equal shares is $(z_i, \ldots, z_n)$, so, (29)

$$\sum_{i \in N} z_i = v(N; N) = ne + \frac{\alpha^2}{4b}$$

I check that this payoff belongs to the γ-core of this game. Which is the same as verifing

$$\sum_{i \in S} z_i \geq v\left(S; \{S, [N \setminus S]\}\right) \forall S \subset N, \text{thus,}$$

$$\sum_{i \in S} z_i \geq |S|e + \frac{\alpha^2}{b(n - |S| + 2)^2}$$

$$|S|e + \frac{|S|\alpha^2}{4bn} \geq |S|e + \frac{\alpha^2}{b(n - |S| + 2)^2}$$

since

$$0 < \frac{1}{(n - |S| + 2)^2} \leq \frac{|S|}{4n} < 1$$

the inequality holds.

Now I verify that the largest coalition in each partition is worse-off relative to $(z_1, \ldots, z_n)$.

11. See proposition 2 in Parkash (2019)

Let $P = \{S_1, \ldots, S_m\} \neq [N], \{N\}$ some partition of N. (30)
Assume that $|S_m| \leq \cdots \leq |S_2| \leq |S_1|$. Then $2 \leq m < n$, and I know that
$\sum_{i=1}^{m} v(S_i; P) < v(N; \{N\})$
which impliest that $v(S_1; P) < \sum_{i \in S_1} z_i$ should hold for this game
Then I have that $v(S_1; P) = |S_1|e + \frac{\alpha^2}{b(m+1)^2}$ and $|S_1| \left[e + \frac{\alpha^2}{4bn} \right] = \sum_{i \in S_1} z_i$
since $2 \leq m < n$ implies that $0 < \frac{1}{(m+1)^2} < \frac{|S_1|}{4n} < 1$ for either $|S_1| \geq m$ or $|S_1| < m$
thus the inequality $v(S_1; P) < \sum_{i \in S_1} z_i$ holds.

So far, I am studying the gains of cooperation that the individuals involved in a problem of common pool resources can obtain through the worth of coalitions. In this setting, the game is symmetric, which means that a natural way of sharing the value of a coalition is just dividing it by the number of its members. Every member of a coalition gets the same share of the worth. In this relation, this game is also such that a coalition with more number of members has lower-per members payoffs in each partition. This implies that given a partition different from the gran coalition, the coalition with more members willing to cooperate may not form, since they notice that their individual payoff is lower than if they were in another coalition or singleton. In the context of common pool resources, this implies that when players form a partition or coalition structure, the largest coalition, which is the one with more players being aware of about needs to cut down on the resource extraction, is paradoxically the coalition less stable; notwithstanding being the coalition with greater value. Consider a case in which a partition consists of two coalitions, one wit $n-1$ players and a singleton coalition. Even when the majority is willing to cooperate, this partition disintegrates. A greater size of a coalition relative to the size of other coalitions in a partition discourages the formation of it in favor of the grand coalition.

Example 2 *Say that* $e = 25$*,* $c = 5$*,* $n = 9$*,the total revenue is given by the same function as example 1. Consider the following partition,*

$$P = \{\{1, 2, 3, 4, 5\}, \{6\}, \{7\}, \{8\}, \{9\}\}$$

Thus, the worth of the five coalitions are the next,

$$v(\{1, 2, 3, 4, 5\}; P) = 161$$

$$v(\{i\}; P) = 61, i \in 6, 7, 8, 9$$

Notice that if the largest partition was shared out by 5*, then each member would get:* $32.2 < 61$.

Moreover, say that player 5 *withdraws from the coalition she belonged to, so a new partition* P' *shapes.*

$$P' = \{\{1,2,3,4\},\{5\},\{6\},\{7\},\{8\},\{9\}\}$$

Under this new configuration, the worth of coalitions in P' *are the following,*

$$v(\{1,2,3,4\};P) \approx 126.44$$
$$v(\{i\};P) \approx 51.44, i \in 5,6,7,8,9$$

Which means that were the worth of the coalition from which player 5 *withdrew to split up into its actual cardinality, every member would get* $\frac{126.44}{4} \approx 31.6$*, which is less than the individual value of* 32.2 *when player* 5 *stays in. Thus the withdrawal of this players affects negatively the worth of the remaining players. However, she gets now* 51.4*, which is greater than the individual value she gains in the original partition* P*. In this sense, partition* P *is even less stable than* P'*.*

In this vein, the γ-core of this game exists —as mentioned above —the equal payoff sharing rule belongs to it, and the largest coalition in any partition is in a worse position relative to it. A new, given the symmetry of the game —which, in passing, is due to the homogeneity of the players—equal sharing rule of the grand coalition is fair and comes up naturally. Each of the players gains the same amount. Under these circumstances, applying this rule boots the players to move towards the grand coalition, since if they abide in any other partition, their cooperative gains will be smaller or equal than that one of that rule. This is in line with empirical studies that show that when the group is relatively homogeneous, the individuals tend towards cooperation in terms of self-governing the resource, Bardhan (1993), Libecap (1994), Lam et al. (1998), Varughese and Ostrom (2001), Bardhan, Dayton-Johnson, et al. (2002).

Example 3 *Going back to the example 2. The equal sharing rule pungles up* $\frac{v(N;\{N\})}{9} = \frac{549}{9} = 61$ *to each individual. Thus there are incentives to form the grand coalition.*

4.4 Coalition Formation and The payoff Sharing Game

On the other hand, in the light of results of Parkash (2019), that the γ-core as a cooperative solution concept can be supported as an equilibrium outcome of the so-called payoff sharing game, which I introduce below. Also the grand coalition is the unique equilibrium outcome if and only if the γ-core is non empty. This is another way of conceiving the formation of coalitions. Since I am interested in understanding this issue in the context of the common pool resources, I explore these results in relation to my problem.

The payoff sharing game is a game in two stages. It is played infinity. The stages are:

- First Stage
  - It begins from the finest partition $[N]$ as the *status quo* and each player announces some nonnegative integer from 0 to n.
- Second State
  - All those players who announced the same positive integer in the first stage form a coalition and may either give effect or dissolve it. All those players who announced 0 remain singletons.
- If the outcome of the of the second stage is not the finest partition, the game ends and the partition formed remains formed forever. But if the outcome in the second stage is the finest partition, the two stages are repeated, until some nontrivial partition is formed in a future period. In either case, the outcome of the second stage is a partition in which players receive payoffs, in each period, in proportion to a pre-specified feasible payoff vector $(z_i^*, \ldots, z_n^*)$

Suppose that the community of n individuals is interested in the preservation and in the moderate extraction of the resource, so they have a meet in order to decide upon how to coordinate and who works with whom knowing in advance what their payoffs will be in each partition. If the players agree to form a partition different from the finest one, the meeting ends. And they get payoffs according a predetermined rule. Otherwise, the meet lasts until a partition different from the finest one takes place. That is to say, the meet comes off with participation through an agreement. Related to this, there are in the field cases where people meet with management and extract a common pool resource. As an example of this, there is the case of the study of indigenous people in Oaxaca (mentioned in the introduction of this work), where under the framework of *usos y costumbres* program, they have meetings on a regular basis to deliberate responsibilities, charges, and duties regarding the extraction and management of their own resources. Thus they form groups of work[12]. In this sense, the payoff sharing game is useful to understand processes of formation of coalitions as in this example. In this game, the specified payoff vector plays a significant role, since the players will anchor their strategies to this. A priori, any partition could be a possible outcome of the second stage.

Parkash (2019) proves specifically that as long as a partition function game is partially super-additive with nonempty γ-core, each payoff vector $(z_1^*, \ldots, z_n^*)$ that belongs to it is actually an equilibrium payoff vector of the payoff sharing game in which payoffs are assigned in proportion to this vector. A partially super-additive partition function means that combining only all non-singleton coalitions in a partition increases their total worth. Formally it is,

12. For instance, young women carry out activities different from those of young man, who typically do the hard work whereas others chose not to be part of it but to make up for it by paying a fine

Definition 5 (Partially Super-additive Partition Function) *A partition function (N, v) is partially super-additive if for any partition $P = \{S, [N \setminus S]\}$ and $\{S_1, \ldots, S_k\}$ such that $\bigcup_{i=1}^{k} S_i = S, |S_i| > 1, i = 1, \ldots, k$, $\sum_{i=1}^{k} v(S_i; P') \leq v(S; P)$, where $P' = P \setminus S \cup \{S_1, \ldots, S_k\}$.*

In order to prove that γ-core payoff vectors can be equilibrium payoff vectors, the author shows that to dissolve a coalition if it does not include all players is an equilibrium strategy of each player, and that the grand coalition N is an equilibrium outcome resulting in per-period equilibrium payoffs equal to $(z_1^*, \ldots, z_n^n)$. Also, he characterizes the equilibrium of the repeated game by comparing per-period payoffs of the players.

So a natural question comes to my mind, what implications would entail for the players involved in a common pool issue to play the payoff sharing game? First of all, this game is a way of incorporating a mechanism of communication, since they have the possibly of forming or not a coalition in the second stage. Allowing for communication might improve results from group interaction, Ostrom, Walker, and Gardner (1992). Second, under "round bargains" their efforts will be in favor of forming the grand coalition. And third, that the coalitions different from it will no be stable in the sense that, it is not an equilibrium strategy for each player to materialize them.

That said, I know that the partition function of the common pool game is symmetric and that the grand coalition is the efficient partition, so its γ-core is nonempty. Next, I have to verify whether it is partially super-additive or not. Since the grand coalition is efficient, for the case of three or and four players partial partial super-additivity holds. For more than four players, let me use the example 2.

Example 4 *Say that $e = 25$, $c = 5$, $n = 9$,the total revenue is given by the same function as example 1. Consider the following partition,*

$$P = \{\{1, 2, 3, 4, 5\}, \{6\}, \{7\}, \{8\}, \{9\}\}$$

Now say that $S = \{1, 2, 3, 4, 5\}$ and that $S_1 = \{1, 2\}$ and that $S_2 = \{3, 4, 5\}$, then $S_1 \cup S_2 = S$ and

$$\begin{aligned} P' &= \{\{P \setminus S\} \cup \{S_1, S_2\}\} \\ P' &= \{[N \setminus S] \cup \{\{1, 2\}, \{3, 4, 5\}\}\} \\ P' &= \{\{1, 2\}, \{3, 4, 5\}, \{6\}, \{7\}, \{8\}, \{9\}\} \end{aligned} \tag{31}$$

The worth of coalitions S, S_1, S_2 are the next,

$$v(\{S\}; P) = 161$$

$$v(\{S_1\}; P') \approx 76.44$$

$$v(\{S_2\};P') \approx 101.44$$

thus

$$v(\{S\};P) < v(\{S_1\};P') + v(\{S_2\};P')$$

Thus the partition function of the common pool game is not partially super additive for cases of more than four players. Which means that in the payoff sharing game when the number of members of a group involved in a common pool issue is relatively small (say three or four) they will end up grouping. The grand coalition is an equilibrium outcome. Moreover, as Parkash (2019) shows, the grand coalition is the only equilibrium outcome if the players believe that the finest partition (every one single) is not a strategically relevant equilibrium outcome. Also when the game is played once, the grand coalition remains as an equilibrium outcome in the case of three players. See the following example.

Example 5 *Say that $n = 3$, player i may consider a deviation of the grand coalition to the partition $P = \{\{i\}\{j,k\}\}$, which will be strategically relevant rather than the finest partition if the payoffs of the other two players are higher in partition P than in the finest one.*

$$v(\{N\};\{N\}) = 3e + \frac{\alpha^2}{4b}$$

$$v(\{i\};\{\{i\},\{j,k\}\}) = e + \frac{\alpha^2}{9b}$$

$$v(\{j,k\};\{\{i\},\{j,k\}\}) = 2e + \frac{\alpha^2}{9b}$$

$$v(\{i\};\{\{i\},\{j\},\{k\}\}) = e + \frac{\alpha^2}{16b}$$

Under this structure, as the game is symmetric, then the feasible payoff vector with equal shares belongs to the γ-core of this game, then it can be the pre-specified payoff vector. Recall that payoffs are assigned in proportion to this vector. Thus,

$$z_i^* = z_j^* = z_k^* = e + \frac{\alpha^2}{12b},$$

and the individual payoffs if partition P is formed are $e + \frac{\alpha^2}{9b}$ for player i, and $e + \frac{\alpha^2}{18b}$ for players j and k. Players j and k have no incentives to deviate from the grand coalition towards coalition P, since they get better payoffs. In contrast, player i finds it atractive to move to partition P, but she knows that for the others it is not. Then the equilibrium outcome is the grand coalition.

That said, what about cases in which the number of involved players are more than four? It it is not clear yet if an element of the γ-core will be an equilibrium payoff vector of the game, since in this case the function is not partially super additive. Also, it may happen that an equilibrium outcome of the game is not necessarily the grand coalition. This motivates the research of more scenarios that capture coalition patterns where intermediate coalition structures come up as equilibrium outcomes.

5 Results

This work consists of three parts. In the first part I establish the current framework of the common pool resources, definition and classification. Also I set out the problem of appropriation and its nature: the conditions under which is its a dilemma or just a situation. Also, there are some common variables that explain cooperative behavior, although some of them are still a matter of debate. Thus I observe that it is a complex, multifaceted issue. Next, typically when the involved individuals face a dilemma, it is studied with game theory, which predicts an non cooperative outcome, so I drew from this. I took the baseline model known as common pool game used in the experiments of Ostrom et al. (1994). In this second part, based on the sense of cooperation that some individuals may have and under the assumption of the model, I study conditions in which forming a group of cooperative members may be beneficial. I found that forming a group smaller than the total population is positive for cooperative people as long as it is not too small relative to the total population, which at the same time should not be too large. When the population gets large, the cooperative group should be relatively large as well. Although, under these circumstances the resource may not resist large populations. Thus in this setting, small groups get better gains. A cooperative group in a big population is hard to sustain. Now in the last part of the work, which I am still working on, I transform the original common pool game into a partition function game. I am studying formation of coalition structures and the gains of cooperation, for which I started out by applying some recent results regarding strategic games in partition form. So far I found that given the symmetry of the original game its partition function version is symmetric and the grand coalition is an efficient partition in the sense that maximizes the total payoff of all players, this two properties are fundamental, since they guarantee that the so called γ-core is not empty, and an element of it is the equal payoff sharing. Also I found that partitions different from the grand coalition partition will not be stable, so the efforts of the players move towards full cooperation. In addition to that, I studied a game of two stages for formation of coalition structures called the payoff sharing game in relation to the γ-core of the common pool game. In this game for cases in which there are three or four players, an equilibrium outcome of the payoff sharing game is the grand coalition. However, for cases of more that four players it is not clear yet what coalition structure may emerge.

6 Discussion

One concern is worthwhile to mention: the endowments. Neither the Nash equilibrium investment appropriation decision nor the social optimum depend on the endowments as long as e is sufficiently large, Ostrom et al. (1994). However, it may happen that small values of the endowments are actually the solution to the maximization problem the individuals face in either the strategic game and/or in the partition function game. I do not consider those cases. On the other hand, the results depend highly on the symmetry assumption of the players. More accurate explanations can be arrived at by changing this assumption. In the case of the partition function form, introducing asymmetry in the endowments and costs may influence the formation of coalition structures other than the grand coalition, since the value of a coalition includes them. When the symmetric assumption of costs and endowments is weaken, their effect on the model is more apparent and can be studied separately.

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A Appendix

A.1 Groups and Individual Behavior

Proof (Proposition 4). The utility of the potential group S of implementing the social optimum while the other the NE is

$$\sum_{i\in S} u_i(x_i^{**}, x_{-i}^{*}) = |S|e + \frac{|S|}{n}\left(\frac{\alpha}{2b}\right)\left[\alpha - b\left[\frac{|S|}{n}\left(\frac{\alpha}{2b}\right) + (n-|S|)\frac{\alpha}{b(n+1)}\right]\right] \tag{32}$$

$$= |S|e + \frac{\alpha^2 |S|\left(n|S| + 2n - |S|\right)}{4b(n+1)n^2}$$

whereas the utility of group S of implementing the NE just like the others is

$$\sum_{i\in S} u_i(x_i^{*}, x_{-i}^{*}) = |S|e + |S|\left(\frac{\alpha}{b(n+1)}\right)\left[\alpha - b\left[\frac{|S|\alpha}{b(n+1)} + \frac{(n-|S|)\alpha}{b(n+1)}\right]\right] \tag{33}$$

$$= |S|e + \frac{|S|}{b}\left[\frac{\alpha}{(n+1)}\right]^2$$

Now the people interested will be indifferent between forming the group or nor when

$$|S|e + \frac{\alpha^2 |S|\left(n|S| + 2n - |S|\right)}{4b(n+1)n^2} = |S|e + \frac{|S|}{b}\left[\frac{\alpha}{(n+1)}\right]^2$$

Solving for $|S|$,

Find Least Common Multiplier of

$4b\,(n+1)\,n^2,\ b: \quad 4n^2b\,(n+1)$

Multiply by LCM $= 4n^2b\,(n+1)$

Simplify (34)

$$4en^2|S|b\,(n+1) + |S|\alpha^2\,(n|S| + 2n - |S|) = 4en^2|S|b\,(n+1) + \frac{4n^2|S|\alpha^2}{n+1}; \quad n \neq 0,\ n \neq -1$$

$$4en^3|S|b + 4en^2|S|b + n|S|^2\alpha^2 + 2n|S|\alpha^2 - |S|^2\alpha^2 = \frac{4n^2|S|\alpha^2 b + 4en^4|S|b + 8en^3|S|b + 4en^2|S|b}{n+1};$$

$n \neq 0,\ n \neq -1$

Subtract $\dfrac{4n^2|S|\alpha^2 + 4en^4|S|b + 8en^3|S|b + 4en^2|S|b}{n+1}$ from both sides

Simplify

$$\left(\frac{n^2\alpha^2}{n+1} - \frac{\alpha^2}{n+1}\right)|S|^2 + \left(-\frac{2n^2\alpha^2}{n+1} + \frac{2n\alpha^2}{n+1}\right)|S| = 0; \quad n \neq 0,\ n \neq -1$$

Solve with the quadratic formula

$$|S| = \frac{-\left(-\frac{2n^2\alpha^2}{n+1} + \frac{2n\alpha^2}{n+1}\right) + \sqrt{\left(-\frac{2n^2\alpha^2}{n+1} + \frac{2n\alpha^2}{n+1}\right)^2 - 4\left(\frac{n^2\alpha^2}{n+1} - \frac{\alpha^2}{n+1}\right)0}}{2\left(\frac{n^2\alpha^2}{n+1} - \frac{\alpha^2}{n+1}\right)} : \quad 0$$

$$|S| = \frac{-\left(-\frac{2n^2\alpha^2}{n+1} + \frac{2n\alpha^2}{n+1}\right) - \sqrt{\left(-\frac{2n^2\alpha^2}{n+1} + \frac{2n\alpha^2}{n+1}\right)^2 - 4\left(\frac{n^2\alpha^2}{n+1} - \frac{\alpha^2}{n+1}\right)0}}{2\left(\frac{n^2\alpha^2}{n+1} - \frac{\alpha^2}{n+1}\right)} : \quad \frac{2n}{n+1}$$

The solutions to the quadratic equation are :

$$|S| = 0,\ |S| = \frac{2n}{n+1}; \quad n \neq \sqrt{1},\ n \neq -\sqrt{1}$$

Now, if I take $|S| > \frac{2n}{n+1}$ the utility (32) of the group S is greater strictly than the utility (33) as long as n does not get large while $|S|$ fixed. Otherwise observe that

$$\lim_{n\to\infty} \textstyle\sum_{i\in S} u_i(x_i^{**}, x_{-i}^*) = \lim_{n\to\infty} \sum_{i\in S} u_i(x_i^*, x_{-i}^*) = |S|e$$

■

Proof (Proposition 5). The group size is $|S|$, which at most could be n itself. Thus $|S| \leq n$, thus

$$
\begin{aligned}
& -n \leq -|S| \\
& |S| - n \leq 0 \\
& (|S| - n)^2 \geq 0 \\
& -(|S| - n)^2 \leq 0 \\
& -|S|^2 + 2n|S| - n^2 \leq 0 \\
& \text{add up } n^2|S|^2 - n^2|S|^2 \text{ to both sides of the inequality} \\
& n^2|S|^2 - |S|^2 + 2n|S| - n^2 \leq n^2|S|^2 \\
& \text{sum up } 2n^2|S| - 2n^2|S| \text{ to both sides of the inequality} \\
& n^2|S|^2 - |S|^2 + 2n^2|S| + 2n|S| - n^2 \leq n^2|S|^2 + 2n^2|S| \\
& n^2|S|^2 - |S|^2 + 2n^2|S| + 2n|S| - n^2 \leq n^2|S|^2 + 2n^2|S| + n^2 - n^2 \\
& n^2|S|^2 - |S|^2 + 2n^2|S| + 2n|S| \leq n^2|S|^2 + 2n^2|S| + n^2 \\
& |S| \left(n^2|S| - |S| + 2n^2 + 2n \right) \leq n^2(|S| + 1)^2 \\
& \text{add up } n|S| - n|S| \text{ to the parentehis of the left side of the inequality} \\
& |S| \left(n^2|S| - |S| + 2n^2 + 2n + n|S| - n|S| \right) \leq n^2(|S| + 1)^2 \\
& |S| \left(n|S| + 2n - |S| \right) (n + 1) \leq n^2 \left(|S| + 1 \right)^2 \\
& \frac{|S| \left(n|S| + 2n - |S| \right)}{n^2} \leq \frac{\left(|S| + 1 \right)^2}{(n + 1)} \\
& \text{Muliply both sides by } \frac{\alpha^2}{4b(n+1)} \text{ so as to get} \\
& \\
& \frac{\alpha^2 |S| \left(n|S| + 2n - |S| \right)}{4b(n+1)n^2} \leq \left(\frac{1}{b} \right) \left[\frac{\alpha(|S| + 1)}{2(n+1)} \right]^2 \\
& \text{Add } |S|e \text{ to both sides} \\
& \\
& |S|e + \frac{\alpha^2 |S| \left(n|S| + 2n - |S| \right)}{4b(n+1)n^2} \leq |S|e + \left(\frac{1}{b} \right) \left[\frac{\alpha(|S| + 1)}{2(n+1)} \right]^2 \\
& \text{so it is actually the inequality} \\
& \sum_{i \in S} u_i(x_i^{**}, x_{j \notin S}^*) \leq \sum_{i \in S} u_i(x_S^*, x_{j \notin S}^*)
\end{aligned}
$$

(35)

(36)

meting the equality when $|S| = n$, or whenever $|S|$ approaches to a given n not too large, since

$$\lim_{|S|\to n}\left(|S|e+\frac{1}{b}\left(\frac{\alpha\left(|S|+1\right)}{2\left(n+1\right)}\right)^2\right)=\lim_{|S|\to n}\left(|S|e+\frac{\alpha^2|S|\left(n|S|+2n-|S|\right)}{4b\left(n+1\right)n^2}\right)=en+\frac{\alpha^2}{4b}.$$

Now for a given $|S|$, as n is increasingly large, I have that

$$\lim_{n\to\infty}\sum_{i\in S}u_i(x_i^{**},x_{j\notin S}^*)=\lim_{n\to\infty}\sum_{i\in S}u_i(x_S^*,x_{j\notin S}^*)=|S|e,$$

but if $|S|\to n\to\infty$ thus

$$\lim_{|S|\to n\to\infty}\left(|S|e+\frac{1}{b}\left(\frac{\alpha\left(|S|+1\right)}{2\left(n+1\right)}\right)^2\right)=\lim_{|S|\to n\to\infty}\left(|S|e+\frac{\alpha^2|S|\left(n|S|+2n-|S|\right)}{4b\left(n+1\right)n^2}\right)=\infty$$

■